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1

LIST OF THE COMBINATORIAL STRUCTURES FROM DLSS OF ORDER 10 ON ORTHOGONALITY RELATIONSHIP SET

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Basic definitions

DLS – diagonal Latin square (in this document of order 10).

Orthogonal DLSs (ODLS) – pair of DLS *A* and *B*, in which all ordered pairs of elements $\left(a_{ij}, b_{ij}\right)$ are distinct.

String representation of DLS – elements of DLS that are written from left to right from top to bottom:

```
3 2 8 4 6 7 1 0 9 5
8 1 2 7 4 6 3 5 0 9
1 5 0 9 8 2 4 3 7 6
6 8 5 2 0 9 7 1 4 3
9 0 7 1 5 4 2 6 3 8
4 3 9 0 1 8 6 7 5 2
0 6 3 8 7 5 9 2 1 4
5 7 6 3 9 1 8 4 2 0
7 9 4 5 2 3 0 8 6 1
2 4 1 6 3 0 5 9 8 7
```

3284671095812746350915098243766852097143907154263843901867520638759214576391842079452308612416305987

DLX – Dancing Links X algorithm – algorithm for exact cover problem solving.

Canonical form (CF) – lexicographically minimal string representation of DLS within corresponding isomorphism (isotopism) class.

Symmetry – some correspondence between elements of DLS. Can be viewed geometrically (vertical or horizontal plane symmetry, central symmetry) or generalized through structure of lengths of loops of permutations $\left(P_X, P_Y, P_V\right)$. For $N = 10$ well known $10! = 3\,628\,800$ different permutations but only 42 from them has different structures of loops.

Partially symmetric DLS – DLS, in which only *M* cells corresponds to some symmetry and another cells has asymmetric filling, in correspondence with definition of correct DLS.

Combinatorial structure – graph, in which vertices are DLSs and edges are corresponds to the orthogonality binary relationship between pair of DLSs. The set of different combinatorial structures that are found within distributed computing project Gerasim@Home is shown below.

ESODLS (Extended SODLS) – DLS, that has ODLS from the same main class. Finding of such ODLS pairs can be effectively organized without transversals using CMS.

CMS – cells mapping schema for DLS. It can be used to search for ODLS without transversals.

1. DLS without ODLS (bachelor)

a) A   b) 1

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789120453986735618740929478321605273098514668592473104697013258704619852383156029745982760431 (DLS A, CF 1).

Adjacency matrix:

$M = (0)$.

Different CFs set within combinatorial structure:

CF 1: 0123456789120453986735618740929478321605273098514668592473104697013258704619852383156029745982760431 (DLS A).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 1, a = 0, \rho = [1]$.

Method of finding:

any type of search due to most amount of DLSs hasn't ODLS.

2.1. ODLS pair without symmetry (asymmetric once)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204365978201563489734579816205749802163698124730583760 19254963072854175681904324892573016 (DLS *A*, CF 1);

DLS 2: 01234567893758692410498710325612097458639412580637209436857168302 71945854601739256719340287365829104 (DLS *B*, CF 2).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204365978201563489734579816205749802163698124730583760 19254963072854175681904324892573016 (DLS *A*);

CF 2: 01234567891230679845897406135237592480164568123907760258419398473 15260241593067850867924316391807524 (DLS *B*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 2,\ a = 1,\ \rho = \begin{bmatrix} 1, 1 \end{bmatrix}.$$

Method of finding:

any type of search in combination with method of checking DLS for ODLS (Euler-Parker method, DLX [9]), approximately among 30 millions of DLS exists only 1 DLS with ODLS.

2.2. ODLS pair with symmetry (symmetric once)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204587936543789162067981302549375628041894071356236512
49807708296541325693041784816072395 (DLS *A*, CF 1);

DLS 2: 01234567899357860421256107489346092175383786542910729438510689457
01362581063927410789236456432198057 (DLS *B*, CF 1).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204587936543789162067981302549375628041894071356236512 49
80770829654132569304178481607 2395 (DLS *A*, DLS *B*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 2, a = 1, \rho = [1, 1].$

Method of finding:

Brute Force for self-orthogonal ODLSs (SODLS) [1, 2] (symmetry provided by DLS transposing from the main diagonal or main anti-diagonal); also found different ODLS with same property for different *M*-transformations [3].

3.1. Line-3 without symmetry (asymmetric twice)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204369857278693510480571923463869571420597104863276482 13095453062791894157802636392804571 (DLS *B*, CF 1);

DLS 2: 01234567899652017438527834901638195246071905268374603478259143918 75260748693012585671039422740691853 (DLS *A*, CF 2);

DLS 3: 01234567899652817430527034981638195246071985260374603478259143910 75268740693812585671039422748691053 (DLS *C*, CF 3).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204369857278693510480571923463869571420597104863276482 13095453062791894157802636392804571 (DLS *B*);

CF 2: 01234567891204789356437680129550892176432517648930769813540239605748 21873192056498453620176452093178 (DLS *A*);

CF 3: 01234567891204789356437980126550862179432517948630769813540239605748 21873162059498453620176452093178 (DLS *C*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 3,\ a = 2,\ \rho = \begin{bmatrix} 1, 1, 2 \end{bmatrix}.$$

Method of finding:

any type of search in combination with method of checking DLS for ODLS (Euler-Parker method, DLX), approximately for 500–1000 onces it is present 1 asymmetric twice.

Remark:

line-3 structure also can be treated as «1:2 structure» or as rhombus-1.

3.2. Line-3 with symmetry (symmetric twice)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204365978203581469746971820359786543120347809125669417
28503785063941253192708648562907341 (DLS *B*, CF 1) – horizontal symmetry;

DLS 2: 01234567893857290416479836152092865031746510724893564281903783049
75261246918730510756329487931048652 (DLS *A*, CF 2);

DLS 3: 01234567893859072416974836102552869431706015729843269081753483742
05961496218035715076342987431598602 (DLS *C*, CF 2).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204365978203581469746971820359786543120347809125669417 28
5037850639412531927086485629073 41 (DLS *B*);

CF 2: 01234567891204368957571690384238975240166459810273937824156085601 79
3244932785601264509713870816324 95 (DLS *A*, DLS *C*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 3,\ a = 2,\ \rho = [1, 1, 2].$$

Method of finding:

any type of symmetric DLS search (plane symmetry [4–6], generalized symmetry [7]) in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

4.1. Line-4 without symmetry

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789120457983654682031972675138940678932405138107952647946081325935786041280916425734532917608 (DLS B, CF 1);

DLS 2: 0123456789731594862036597840124298601573987216340550863729411407239856694051723825348901678761025394 (DLS A, CF 2);

DLS 3: 0123456789731598462036597480124298601573947216380550863729411807239456694051723825348901678761025394 (DLS C, CF 3);

DLS 4: 0123456789120453987674682015932615378940658972403138701952645946083127973186045280976423154352917608 (DLS D, CF 4).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456789120457983654682031972675138940678932405138107952647946081325935786041280916425734532917608 (DLS B);

CF 2: 0123456789123069584760578431925419082376284170963579683105244796238051937256140835841279608605974213 (DLS A);

CF 3: 0123456789120456893735719248069486307152593281067487690423156895173420704829156346107352982357689041 (DLS C);

CF 4: 0123456789120453896739872015468015947623749261503858760932146538724190964187230543501698722769380451 (DLS D).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 4, a = 3, \rho = [1, 1, 2, 2]$.

Method of finding:

search for partially generalized symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

4.2. Line-4 with symmetry

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789123867094579640835124619702853380251946785973412062450867391908623517457419286306375194028 (DLS *A*, CF 1);

DLS 2: 0123456789769214835014367908259085327614576423109839706851428341579206250891346762198045734857062931 (DLS *B*, CF 2) – SODLS;

DLS 3: 0123456789123869047597640835124619702853380251496785973412062950867341708623519454719286306345179028 (DLS *C*, CF 2) – SODLS;

DLS 4: 0123456789765914832014367208952085397614976423105839706851428341579206520891346765128049734897062531 (DLS *D*, CF 1).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456789120453897626897051437836124095675098132484950176324972863501356827941093176402585041392867 (DLS *B*, DLS *C*);

CF 2: 0123456789123867094579640835124619702853380251946785973412062450867391908623517457419286306375194028 (DLS *A*, DLS *D*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$$v = 4, \, a = 3, \, \rho = \left[1, 1, 2, 2 \right].$$

Method of finding:

Brute Force of the SODLS.

5. Line-5 with symmetry

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789120436597823491805678765904321948263715058302796147698541032351709284640768132956951728403 (DLS C, CF 3) – horizontal symmetry;

DLS 2: 0123456789357809214672659043184081237695693478520187591204639342618570140687395226105498375897361024 (DLS D, CF 2);

DLS 3: 0123456789358709124618659043724032678195897412560363597804219241837560740621395826105498375798362014 (DLS B, CF 2);

DLS 4: 0123456789120436597823491805678765904321947263815058302796147698541032351709284640867132956951827403 (DLS E, CF 1);

DLS 5: 0123456789120436597823491805678765904321948163725058302796147698541032351709284640768231956952718403 (DLS A, CF 1).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456789120436597823491805678765904321947263815058302796147698541032351709284640867132956951827403 (DLS A, DLS E);

CF 2: 0123456789120479563834968120576075184392285736941085602739414638901275731954082697410285635982637104 (DLS B, DLS D);

CF 3: 0123456789120436597823491805678765904321948263715058302796147698541032351709284640768132956951728403 (DLS C).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 5, \; a = 4, \; \rho = \begin{bmatrix} 1, 1, 2, 2, 2 \end{bmatrix}.$$

Method of finding:

Brute Force for horizontally symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

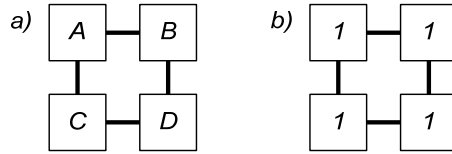## 6.1. Loop-4 (1 different CF)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891206798345367984251040852371965948360271786401395264521
79038951760482323905814678731925604 (DLS *A*, CF 1);

DLS 2: 01234567895389071426159263780496741852307450213968324876901527069
48153693182054780153946724867502391 (DLS *B*, CF 1);

DLS 3: 01234567898359071426159263780496741852307480213965324576901827069
48153693182054750183946724867502391 (DLS *C*, CF 1);

DLS 4: 01234567891206798345363798425104085267193594863027178640139523 4521
79068951730482626905814378731925604 (DLS *D*, CF 1).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891206798345367984251040852371965948360271786401395264521 79
03895176048232390581467873192560 4 (DLS *A*, DLS *B*, DLS *C*, DLS *D*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 4, a = 4, \rho = [2, 2, 2, 2].$$

Method of finding:

- search for partially symmetric DLSs («+–»-symmetry, symmetric filled cells number $M = 70$ [7]) in combination with method of checking DLS for ODLS ODLS (Euler-Parker method, DLX);
- search for ESODLS using CMS without transversals [10].

Remark:

loop-4 structure also can be treated as rhombus-2.

6.2. Loop-4 (2 different CFs)

a)

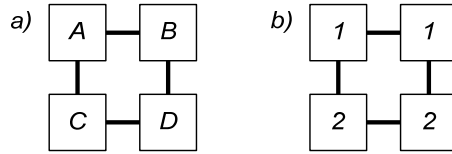| A | — | B |
| C | — | D |

b)

| 1 | — | 1 |
| 2 | — | 2 |

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567893296017845783069512493472086512685930417197854320645627
81093840136957257198243606054172938 (DLS A, CF 1) – SODLS;

DLS 2: 01234567895471938620431672905819028745366094382175853016794292870
15364275964081376485032913865291407 (DLS B, CF 1) – SODLS;

DLS 3: 01234567895417938620437612905879028145366094382175853076194292810
75364275964081316485032973865297401 (DLS C, CF 2) – SODLS;

DLS 4: 01234567898296017345783069512493472086512685930417197854320645627
81093340186957257193248606054172938 (DLS D, CF 2) – SODLS.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891206578934609473251897452618034539807126598061347273581246
9086729403513417089265286139504/ (DLS A, DLS B);

CF 2: 01234567891204678953951624307836871942056748930512409578263179310258
6484695013275872369140235081749/6 (DLS C, DLS D).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 4$, $a = 4$, $\rho = [2, 2, 2, 2]$.

Method of finding:

Brute Force of SODLS.

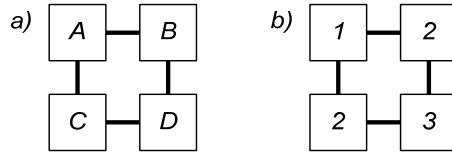## 6.3. Loop-4 (3 different CFs)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789120463597820351846973847902516798654310264798102538560729341461209783593582714605791368024 (DLS *A*, CF 1) − horizontal symmetry;

DLS 2: 0123456789893726450175896401321064725398345018927642983710659876532410534190862767028139542615097843 (DLS *B*, CF 2);

DLS 3: 0123456789894537260176895301421064725398327018945643982610759857643210273190856454068179236512094837 (DLS *C*, CF 2);

DLS 4: 0123456789120463597820358146973847902516791654380264791802538560729341468209713593512784605798361024 (DLS *D*, CF 3) − horizontal symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456789120463597820351846973847902516798654310264798102538560729341461209783593582714605791368024 (DLS *A*);

CF 2: 0123456789123058796434791608525861972043960483521769520483718097613425231679450875483216904785209136 (DLS *B*, DLS *C*);

CF 3: 0123456789120463597820315486975698271034486072931583520974617915364802658790214394761832503749810526 (DLS *D*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 4, a = 4, \rho = [2, 2, 2, 2].$$

Method of finding:

Brute Force for horizontally symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

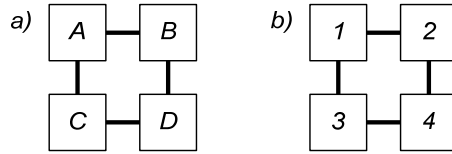## 6.4. Loop-4 (4 different CFs)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:
DLS 1: 0123456789120437956878952430168056792341468753190269328154709761084253357916082424189076355340628197 (DLS A, CF 1);
DLS 2: 0123456789247659830182501796343947615028936182745050897621436514203897473208196578953402161608934572 (DLS B, CF 2);
DLS 3: 0123456789247659830182501796343947615028936182745050897621437514203896463208197568953402171708934562 (DLS C, CF 3);
DLS 4: 0123456789120437956878952430168056792341461753890269321854709768014253357986012424819076355340621897 (DLS D, CF 4).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:
CF 1: 0123456789120437956878952430168056792341468753190269328154709761084253357916082424189076355340628197 (DLS A);
CF 2: 0123456789120453896726709851345489270613395682704165987134209015364872784160239543670912588732149506 (DLS B);
CF 3: 0123456789120497863567308251945617392048437150982639426875109068714352285604197384951302677589263401 (DLS C);
CF 4: 0123456789120437956878952430168056792341461753890269321854709768014253357986012424819076355340621897 (DLS D).

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$v = 4,\ a = 4,\ \rho = [2, 2, 2, 2].$

Method of finding:
search for partially generalized symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

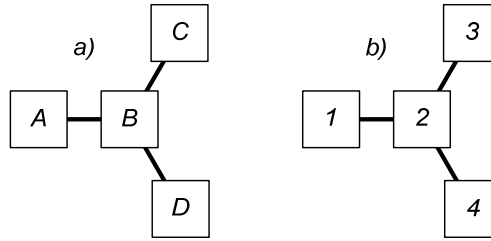## 7. Treshka (1:3 structure)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891237098546409637182596845130725968702431345928061787016
3529423759641087512849360684012 7953 (DLS B, CF 1);

DLS 2: 01234567896951780432924016857335769028418639514207481762395013648
97025578204139620983756147405239168 (DLS A, CF 2);

DLS 3: 01234567899304872165125893704684970653212075648913453671920879621
8453067192038545840321697368 1590472 (DLS C, CF 3);

DLS 4: 01234567899374802165125893704684970653212705648913453671920879621
8453060192738545840321697368159047 2 (DLS D, CF 4).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891237098546409637182596845130725968702431345928061787016 35
2942375964108751284936068401 27953 (DLS A);

CF 2: 01234567891204538967608529731445169820733492715608763984015229601 73
84598476215308751304296537806 9421 (DLS B);

CF 3: 01234567891204687935936250817437910642588549721306543789061269801 35
42740562738912678319540781594 2063 (DLS C);

CF 4: 01234567891204687935956230817437910642588349721506543789061269801 35
4274056273891267851934078159 42063 (DLS D).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 4, \, a = 3, \, \rho = \begin{bmatrix} 1, 1, 1, 3 \end{bmatrix}.$$

Method of finding:

horizontally symmetric DLSs canonization, search for partially generalized symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
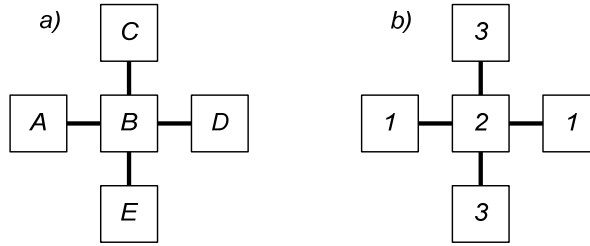
8.1. Symmetric four (1:4 structure)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 012345678912043659782351098467859672304137491805264068271395748069
39152691254780396758142305837902614 (DLS *B*, CF 1) – horizontal symmetry;

DLS 2: 012345678948502796136037182594961254783082659034711704365928397681
4205538972014674916380522548091367 (DLS *A*, CF 2);

DLS 3: 012345678948592706136937182504961254783082659034711794365028307681
4295538072914674016389522548091367 (DLS *C*, CF 3);

DLS 4: 012345678968302794155047182693961254783082569043711704365928497581
3206358972016474916380522368091547 (DLS *D*, CF 2);

DLS 5: 012345678968392704155947182603961254783082569043711794365028407581
3296358072916474016389522368091547 (DLS *E*, CF 3).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 012345678912043659782351098467859672304137491805264068271395748069
152691254780396758142305837902614 (DLS *B*);

CF 2: 0123456789120467983530981645272836791054795132840693158472604780935
612867950214354620839716547210398 (DLS *A*, DLS *D*);

CF 3: 0123456789120457896390316278544765982031389671524079183604252380194
576657984310284520316975647209318 (DLS *C*, DLS *E*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 5, \, a = 4, \, \rho = \left[1, 1, 1, 1, 4\right].$$

Method of finding:

search for horizontally symmetric and partially generalized symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

Remark:

first time horizontally symmetric string-inverse DLS corresponding for 1:4 structure was described in [4], but this article has only 3 ODLS for them; 4-th ODLS was found and described in article [8].
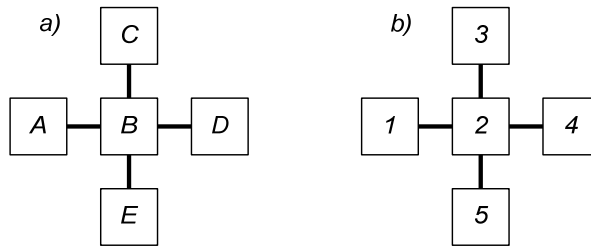
8.2. Asymmetric four (1:4 structure)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204379568348509217657392648019561708243784651039260529
83417297084163543186279508697135024 (DLS *A*, CF 1);

DLS 2: 01234567899546287031603754981243587126903602974158278516390478906
21345826109547314798305265914308267 (DLS *B*, CF 2);

DLS 3: 01234567899546187032603754982143587216903602974158178526390478906
12345826109547324798305165914308267 (DLS *C*, CF 3);

DLS 4: 01234567894596287031603754981293587126403602974158278516390478406
21395826109547314798305265914308267 (DLS *D*, CF 4);

DLS 5: 01234567894596187032603754982193587216403602974158178526390478406
12395826109547324798305165914308267 (DLS *E*, CF 5).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204379568348509217657392648019561708243784651039260529 83
41729708416354318627950 8697135024 (DLS *A*);

CF 2: 01234567891204537968468729350198156724306092315847857106429354 38709
61237469801252359148076 7960821354 (DLS *B*);

CF 3: 01234567891204538967275869034196872145306435179802394608 72154892703
15650198426737560321498 8371965024 (DLS *C*);

CF 4: 01234567891204587963468729350198156724306092315847357106429854 38709
61287469301252359148076 7960821354 (DLS *D*);

CF 5: 01234567891204538967275869034198672145306435179802394608 72154692703
15850198426737580321496 8371965024 (DLS *E*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 5, \, a = 4, \, \rho = [1, 1, 1, 1, 4].$$

Method of finding:

search for partially generalized symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
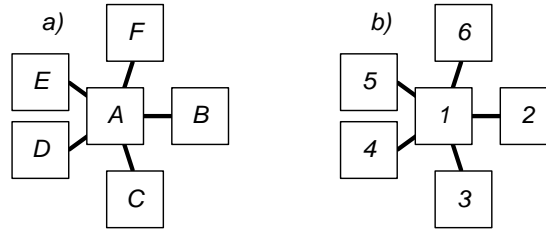
9. Five (1:5 structure)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204879563931768042537950482162586937104794826135086315
04972647032589140597126385862193047 (DLS A, CF 1);

DLS 2: 01234567898596034271673259184054897120637864105932925738061419456
27308201896345736708491254301278596 (DLS B, CF 2);

DLS 3: 01234567893596084271678259134054397120687864105932925783061419456
27803201896345786703491254301278596 (DLS C, CF 3);

DLS 4: 01234567893596084271678259314054197320687864305912925781063419456
27803203896145786701493254301278596 (DLS D, CF 4);

DLS 5: 01234567893796084251658279134074593120685864103972927583061419476
25803201896743586305491274301278596 (DLS E, CF 5);

DLS 6: 01234567893796084251658279314074591320685864301972927581063419476
25803203896741586105493274301278596 (DLS F, CF 6).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204879563931768042537950482162586937104794826135086315049
7264703258914059712638586219047 (DLS A);

CF 2: 01234567891230675948968210435785740321967495381620481659307239618
47205230796851457492108636058729431 (DLS B);

CF 3: 01234567891230675948968710435285740321962495381670481659302739618
47205730296851457492108636058729431 (DLS C);

CF 4: 01234567891204537968397086512424896705139067324851564871329063951
08472853124960778520913464716982035 (DLS D);

CF 5: 01234567891206578934403726589164819273509358710426387904261556148
09273796018354285423910672795634108 (DLS E);

CF 6: 01234567891204587963306274589196478013255389172640651839027448709
23516795106843284362190572795634108 (DLS F).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 6$, $a = 5$, $\rho = [1, 1, 1, 1, 1, 5]$.

Method of finding:

search for partially central symmetric DLSs, symmetric filled cells number $M = 60$, in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
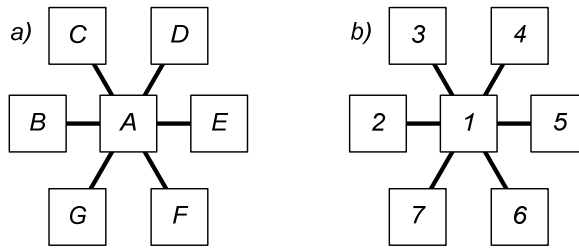
10.1. Asymmetric six (1:6 structure)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567896582097413289163504773085621948917340256405987136232769
14508143528967097641038255640728931 (DLS *A*, CF 1);

DLS 2: 01234567891206783945534890726146398215077562138094291756043884906
75312907431285638512496706785094123 (DLS *B*, CF 2);

DLS 3: 01234567891806723945534290786146392815077568132094291756043884906
75312907431825632518496706785094123 (DLS *C*, CF 3);

DLS 4: 01234567891806273945534790286146397815022568137094791256043884906
25317907431825632518496706785094123 (DLS *D*, CF 4);

DLS 5: 01234567891206789345594830726146918235077562918034231756049884306
75912907413285638592416706785094123 (DLS *E*, CF 5);

DLS 6: 01234567891806729345594230786146912835077568912034231756049884306
75912907413825632598416706785094123 (DLS *F*, CF 6);

DLS 7: 01234567891806279345594730286146917835022568917034731256049884306
25917907413825632598416706785094123 (DLS *G*, CF 7).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891234068975269538401764875312907950842136407621985337186954029561720348834290756158091736 24 (DLS *A*);

CF 2: 01234567891206783945534890726146398215077562138094291756043884904 6753129074312856385124967067850940123 (DLS *B*);

CF 3: 01234567891234509867795186240343687210953079684152249031567868429375108506273941978514023656170983 24 (DLS *C*);

CF 4: 01234567891234509867794186250353687210943079684152249031567868529374108506273941978514023646170983 25 (DLS *D*);

CF 5: 01234567891206789345594830726146918235077562918034231756049884306759129074132856385924167067850940 123 (DLS *E*);

CF 6: 01234567891234709865705236149847965280313970815642861927450395476831202408137956638594021758610923 74 (DLS *F*);

<u>CF 7:</u> 012345678912307849564795802613541839706223846195079671048235706213549885479613206809523174395627 0841 (DLS *G*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$$v = 7, a = 6, \rho = \left[1, 1, 1, 1, 1, 1, 6\right].$$

Method of finding:
    search for partially central symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
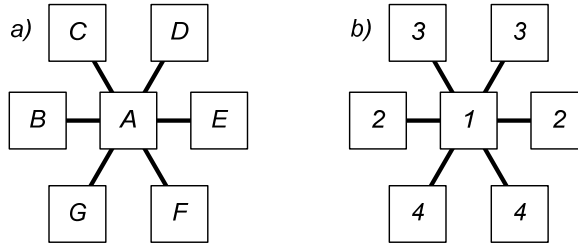
10.2. Symmetric six (1:6 structure)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891234095678234081956798765432108765904321649127805335087 21946501763289476591804324982367105 (DLS *A*, CF 1) − horizontal symmetry, string-inverse square;

DLS 2: 01234567898465172930609723815427518640931649320875321458960753860 97412490271356898706453217538901246 (DLS *B*, CF 2);

DLS 3: 01234567896512309847829504731623047156981970628435563718492098465 73102478196025330682915747459832061 (DLS *C*, CF 3);

DLS 4: 01234567892510967843386259407110348259674651739208970518263479862 43510647930812552480713968397610452 (DLS *D*, CF 3);

DLS 5: 01234567899607284351548167209360953184274219760538293014587678520 93164134682790587645392103578901642 (DLS *E*, CF 2);

DLS 6: 01234567897901384526857219063423549081675087612943961853740242608 75391184906327567352498103496721058 (DLS *F*, CF 4);

DLS 7: 01234567893745168902563908724123819054676507832194795264183080642 19375427639051898105746231498723056 (DLS *G*, CF 4).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891234095678234081956798765432108765904321649127805335087 21946501763289476591804324982367105 (DLS *A*);

CF 2: 01234567891204538967437180925669480753218652947013753916240898106235745486790132279538164030 67214895 (DLS *B*, DLS *E*);

CF 3: 01234567891234089576485192036750768314922497103658398067521496127480356348517920750936284187 65294103 (DLS *C*, DLS *D*);

CF 4: 01234567891204538967249680517338651720947512389640594876130296302478518751920436638709421540 79613528 (DLS *F*, DLS *G*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 7, a = 6, \rho = \left[1, 1, 1, 1, 1, 1, 6\right].$$

Method of finding:

Brute Force for horizontally symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
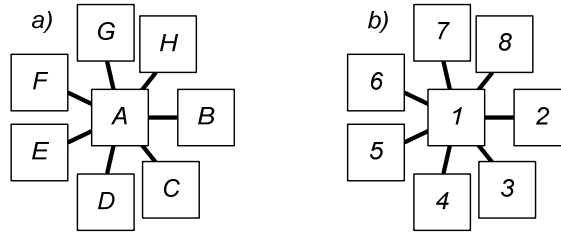
11. Seven (1:7 structure)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789120437956839407158269031847652589762014375680924316475
1382902619584307835296107447862039 15 (DLS *A*, CF 1);

DLS 2: 0123456789396754180286590723142418705936457218369090862371457231
9640585340698271179482056368053194 27 (DLS *B*, CF 2);

DLS 3: 0123456789396154280780592673147418625930154078329696821340752734
90165853760984214297810563680537914 2 (DLS *C*, CF 3);

DLS 4: 0123456789396154280780529673147418695230154078392626891340759734
20165853760284914297810563680537914 2 (DLS *D*, CF 4);

DLS 5: 0123456789396154280780529673147418695230154078392662891340759734
20165853760284914697810523280537914 6 (DLS *E*, CF 5);

DLS 6: 0123456789396154280780592673147418625930154078329696821340752734
09165853769084214207819563689537014 2 (DLS *F*, CF 6);

DLS 7: 0123456789396154280786590273147418205936154278369090861342752734
96105853706984214297810563680537914 2 (DLS *G*, CF 7);

DLS 8: 0123456789396154280786520973147418905236154978362020861349759734
26105853706284914297810563680537914 2 (DLS *H*, CF 8).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456789120437956839407158269031847652589762014375680924316475138
2902619584307835296107447862039 15 (DLS *A*);

CF 2: 0123456789123058964738927650144586271930741832059697046132585967804
3212659147803634509817280719324 65 (DLS *B*);

CF 3: 0123456789120459786394603281752839670514398674205187452196304698135
2076571804392735208194650179634 28 (DLS *C*);

CF 4: 0123456789123460985735879124062906341578841973526048502976139378564
0215762083194604517893276918203 45 (DLS *D*);

CF 5: 0123456789123708964547506913282346578091796482513090827134568591364
2076805237914361894057254791028 63 (DLS *E*);

<u>CF 6:</u> 01234567891204798365359786104290863475217958124630483057921654612039786742015893231968045786759321 04 (DLS *F*);
<u>CF 7:</u> 01234567891204587963946032517828396705143586742091574821963046951382076971804352735209184680179634 25 (DLS *G*);
<u>CF 8:</u> 01234567891206548973473809216559173648028654139027947268053138902754167385921640654170329820698173 54 (DLS *H*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$$v = 8, a = 7, \rho = [1, 1, 1, 1, 1, 1, 1, 7].$$

Method of finding:
search for partially horizontally symmetric DLSs, symmetric filled cells number $M = 60$, in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
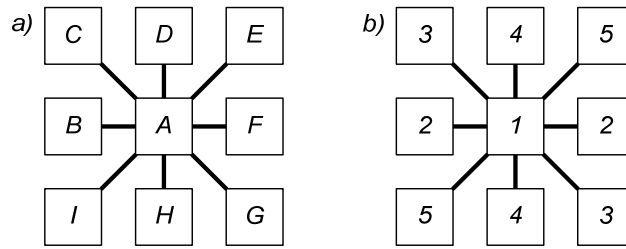
12.1. Symmetric eight (1:8 structure)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 012345678912305496784968271305687490521373591804629487632150359172804687460935215012367894260581493 7 (DLS *A*, CF 1) − horizontal symmetry;

DLS 2: 012345678979013685423476189250853269740117605349282895043617625890137440192758639647812035538472019 6 (DLS *B*, CF 2);

DLS 3: 012345678929013685473476189250853269740117605349287895043612625890137440197258639647812035538427019 6 (DLS *C*, CF 3);

DLS 4: 012345678979013685423476819250153269740887605349212895043617625890137440192758639647182035538472019 6 (DLS *D*, CF 4);

DLS 5: 012345678929013685473476819250153269740887605349217895043612625890137440197258639647182035538427019 6 (DLS *E*, CF 5);

DLS 6: 012345678975413689029470183256895203764117056493282836594017526890147363142708954697812530308972516 4 (DLS *F*, CF 2);

DLS 7: 012345678925413689079470183256895203764117056493287836594012526890147363147208954697812530308927516 4 (DLS *G*, CF 3);

DLS 8: 012345678975413689029470813256195203764887056493212836594017526890147363142708954697182530308972516 4 (DLS *H*, CF 4);

DLS 9: 012345678925413689079470813256195203764887056493217836594012526890147363147208954697182530308927516 4 (DLS *I*, CF 5).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 012345678912305496784968271305687490521373591804629487632150359172804687460935215012367894260581493 7 (DLS *A*);

CF 2: 012345678912043679588960275341983752061476591824302498631507358290417660157498235746813092437109826 5 (DLS *B*, DLS *F*);

CF 3: 012345678912046379588930275641986752031473591824602498361507358290417660157498235746813092467109823 5 (DLS *C*, DLS *G*);

26

CF 4: 01234567891234095876648790125376058349219512783640479816203538502194678961547302234967051850763281 94 (DLS *D*, DLS *H*);

CF 5: 01234567891234095876658790124376058349219412783650479816203538402195678961547302235967041850763281 94 (DLS *E,* DLS *I*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$$v = 9, a = 8, \rho = [1, 1, 1, 1, 1, 1, 1, 1, 8].$$

Method of finding:
Brute Force for horizontally symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
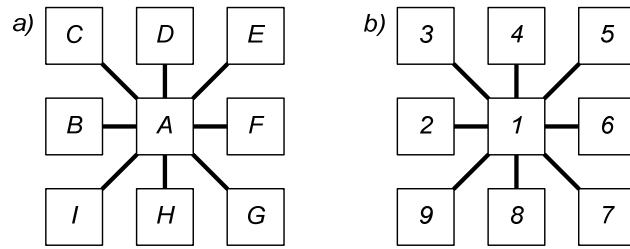
12.2. Asymmetric eight (1:8 structure)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789120457396857321948063018945672894671025348513620977560
829134937560842164892375102697081345 (DLS *A*, CF 1);

DLS 2: 01234567892385697041106483957295471803266418572930397021546842913
08657783906421556027418938756923104 (DLS *B*, CF 2);

DLS 3: 01234567892385967041106483957295471803266418572930397021546842913
08657783609421556027418938759623104 (DLS *C*, CF 3);

DLS 4: 01234567892385697014406183957295471803266418572930397024516812943
08657783906124556027148938756923401 (DLS *D*, CF 4);

DLS 5: 01234567892385967014406183957295471803266418572930397024516812943
08657783609124556027148938759623401 (DLS *E*, CF 5);

DLS 6: 012345678923856970411864039572954718032664105729388397821546042913
08657703986421556027418938756923104 (DLS *F*, CF 6);

DLS 7: 012345678923859670411864039572954718032664105729388397821546042913
08657703689421556027418938759623104 (DLS *G*, CF 7);

DLS 8: 012345678923856970144861039572954718032664105729388397824516012943
08657703986124556027148938756923401 (DLS *H*, CF 8);

DLS 9: 012345678923859670144861039572954718032664105729388397824516012943
08657703689124556027148938759623401 (DLS *I*, CF 9).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456789120457396857321948063018945672894671025348513620977560829
134937560842164892375102697081345 (DLS *A*);

CF 2: 0123456789120463597878395406124391728065256890314769471825035612097
834875036942194862713503075814296 (DLS *B*);

CF 3: 0123456789120463597878395406124591728063238690154769471823055612097
834875036942194682731503075814296 (DLS *C*);

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<u>CF 4:</u> 012345678912046359783952867401431927086580457231967460189352567 8901 23498375426102596318047 6781094523 (DLS *D*);

<u>CF 5:</u> 012345678912046359783952867401431927086580657231947640189352547 8901 23698375426102596318047 6781094523 (DLS *E*);

<u>CF 6:</u> 012345678912046359782839540617439127806575689031426947182503561 2097 83487503694219486721350 3075814296 (DLS *F*);

<u>CF 7:</u> 012345678912046359782839540617459127806373869015426947182305561 2097 83487503694219468723150 3075814296 (DLS *G*);

<u>CF 8:</u> 012345678912043689579862175340834709156264195208737058634291267 0849 13539857124065796283014 4531907628 (DLS *H*);

<u>CF 9:</u> 012345678912046359782346908517695872130454971820633519270846786 0349 15287350946219682517430 4071863295 (DLS *I*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$$v = 9, a = 8, \rho = \left[1, 1, 1, 1, 1, 1, 1, 1, 8\right].$$

Method of finding:

   search for partially central symmetric DLSs, symmetric filled cells number $M = 60$, in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
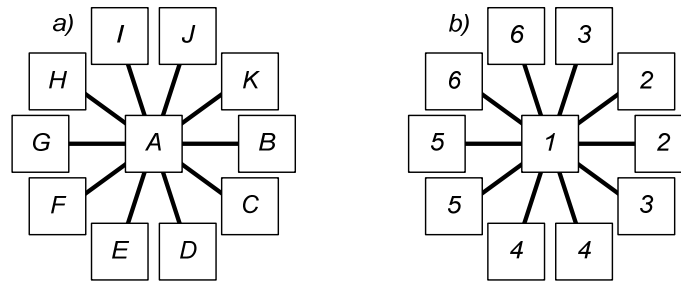
13. Symmetric ten (1:10 structure)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789120487956375429183062781035694863950742194582610375097643812631578294038601942754976320158 (DLS *A*, CF 1) − generalized symmetry (4,31,31);

DLS 2: 0123456789385169420754908726319016248573498571306216375204986702381954854910732623789651407264039815 (DLS *B*, CF 2);

DLS 3: 0123456789485169320753908726419016248573398571406216475203986702381954853910742624789651307264039815 (DLS *C*, CF 3);

DLS 4: 0123456789385169420754901726389016248573498571306286375204916702381954154980732623789651407264039815 (DLS *D*, CF 4);

DLS 5: 0123456789485169320753901726489016248573398571406286475203916702381954153980742624789651307264039815 (DLS *E*, CF 4);

DLS 6: 0123456789385169420754970826319016248573490571386216305274986782301954854917032623789651407264839015 (DLS *F*, CF 5);

DLS 7: 0123456789485169320753970826419016248573390571486216405273986782301954853917042624789651307264839015 (DLS *G*, CF 5);

DLS 8: 0123456789385169420754978026319016248573498571306216305274986702381954854917032623789651407264039815 (DLS *H*, CF 6);

DLS 9: 0123456789485169320753978026419016248573398571406216405273986702381954853917042624789651307264039815 (DLS *I*, CF 6);

DLS 10: 0123456789385169420754971026389016248573498571306286305274916702381954154980732623789651407264039815 (DLS *J*, CF 3);

DLS 11: 0123456789485169320753971026489016248573398571406286405273916702381954153980742624789651307264039815 (DLS *K*, CF 2).

Adjacency matrix:

$$
M = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 01234567891204879563754291830627810356948639507421945826103750976438126315782940386019427549763201 58 (DLS *A*);

<u>CF 2:</u> 01234567891204368957681270934534981725062650917438537984016247362850918947523610908563127475610948 23 (DLS *B*, DLS *K*);

<u>CF 3:</u> 01234567891204379568364918025740589376216385704192291756380498712450365736812940749062831585620914 73 (DLS *C,* DLS *J*);

<u>CF 4:</u> 01234567891204589367635729810479618352404598672031978234061554701638928645701923381902457620369174 58 (DLS *D,* DLS *E*);

<u>CF 5:</u> 01234567891204768935753984016256982743106815923074396701254890465378212350681497478210965384713952 06 (DLS *F,* DLS *G*);

<u>CF 6:</u> 01234567891204589367365709812479618352404598672031978234061554701638928345721906681920457320369174 58 (DLS *H*, DLS *I*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$$v = 11, a = 10, \rho = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 10].$$

Method of finding:

search for generalized symmetric DLSs with (4,31,31) generalized symmetry in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
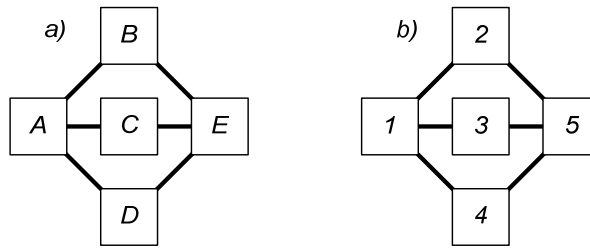
14. Rhombus-3

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891237048965605931247889046215377861935204367528409124908
67153458617932097485036125312790846 (DLS A, CF 1);

DLS 2: 01234567894360295871187296035435418729062937684510509812764396157
48032820951346774560312986784309125 (DLS B, CF 2);

DLS 3: 01234567894860295371137296085425418739068937624510509813764296157
42038320951846774560812936784309125 (DLS C, CF 3);

DLS 4: 01234567894860295371137296085425418739068937642510509813764296157
24038340951826772560814936784309125 (DLS D, CF 4);

DLS 5: 01234567891237048965605931247889045216377861935204357628409124908
67153468517932097486035125312790846 (DLS E, CF 5).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891237048965605931247889046215377861935204367528409124908 67
15345861793209748503612531279 0846 (DLS A);

CF 2: 01234567891204678953698753041248310276952759361804569824307135708 19
24674169825309342705168806519 4327 (DLS B);

CF 3: 01234567891204587936293701564868957304129418623057534987216087612 94
50336701498257586301294405296 8371 (DLS C);

CF 4: 01234567891234069857594681702373082915463759682410801237596448975 23
60195607481326475130298268190 4375 (DLS D);

CF 5: 01234567891236790845674893215048152036973071825964936457120825876 49
03154091683727692084513895031 7426 (DLS E).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 5$, $a = 6$, $\rho = [2, 2, 2, 3, 3]$.

Method of finding:

search for partially central symmetric DLSs, symmetric filled cells number $M = 60$, in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
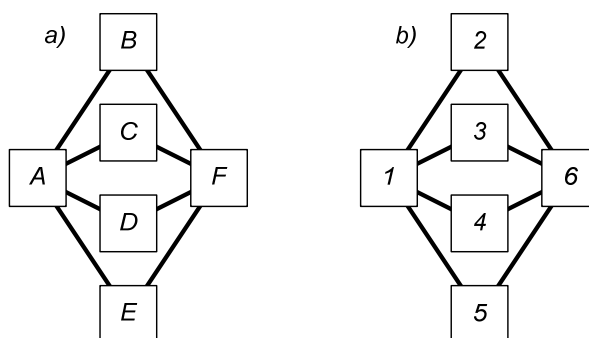
15.1. Rhombus-4 (6 CF)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204368957941827536023579804168960527134584679120360718 43592378910462576950328414532619078 (DLS *A*, CF 1);

DLS 2: 01234567893769081245608594213746928753019548103672741063952859072 64813283451796012763980548351720496 (DLS *B*, CF 2);

DLS 3: 01234567893796081245608594213749628753019548103672741063952856072 94813283451796012793680548351720496 (DLS *C*, CF 3);

DLS 4: 01234567893769081245608594713246928753019548103627241063957859072 64813783451296012763980548351720496 (DLS *D*, CF 4);

DLS 5: 01234567893796081245608594713249628753019548103627241063957856072 94813783451296012793680548351720496 (DLS *E*, CF 5);

DLS 6: 01234567891204568937941827356023579804168960327154384679120560718 45392578910462376950328414532619078 (DLS *F*, CF 6).

Adjacency matrix:

$$
M = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}.
$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204368957941827536023579804168960527134584679120360718 43592378910462576950328414532619078 (DLS *A*);

CF 2: 01234567891204879536604853719297362458013591784260736901842584529630 17587062194326871903544915302678 (DLS *B*);

CF 3: 01234567891206798345506487319274519806239647532801873524901643981672 50387201596425896014376910324578 (DLS *C*);

CF 4: 01234567891234590678476290835183476210959071843562598621743076981352 04280536491735190728466450789123 (DLS *D*);

CF 5: 01234567891230579648305961487289743625014597108326246578019397862430 1568120379547341895260560892143 7 (DLS *E*);

CF 6: 01234567891204568937941827356023579804168960327154384679120560718 4539257891046237695032841453261907 8 (DLS *F*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 6, a = 8, \rho = \left[2, 2, 2, 2, 4, 4\right].$$

Method of finding:

search for partially central symmetric DLSs, symmetric filled cells number $M = 60$, in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
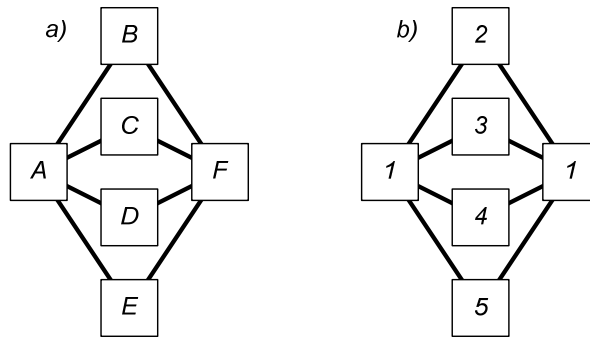
15.2. Rhombus-4 (5 CF)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891234095678341790285659768134029758361240456018932768052 479138649720531708263419523915780 64 (DLS *A*, CF 1);

DLS 2: 01234567893740819526826409537165192708437305184962983754261020516 38497147690325856987210344982367105 (DLS *B*, CF 2) − horizontal symmetry;

DLS 3: 01234567893740819526826409537165197208432305184967983254761070516 38492147690325856982710344987362105 (DLS *C*, CF 3) − horizontal symmetry;

DLS 4: 01234567893750819426826409537164192708537305184962983754261020416 38597157690324856987210344982367105 (DLS *D*, CF 4) − horizontal symmetry;

DLS 5: 01234567893750819426826409537164197208532305184967983254761070416 38592157690324856982710344987362105 (DLS *E*, CF 5) − horizontal symmetry;

DLS 6: 01234567891234095678341790285679568132049578361420276018934568025 749138649720531408563719253912480 67 (DLS *F*, CF 1).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891234095678341790285659768134029758361240456018932768052479 13864972053170826341952391578064 (DLS *A*, DLS *F*);

CF 2: 01234567891204789635957034216829156348705692807341703826591483470 91 2563756918402486152309764891705 23 (DLS *B*);

CF 3: 01234567891204365978378509412653798102649641728530286054931779586 31 40240321876958496273051651790 2843 (DLS *C*);

CF 4: 01234567891204789635957034816289156342705692807341703826591423470 91 856375691240848615230976489170523 (DLS *D*);

CF 5: 01234567891204635978378509412656798102349341728560286054931779583 61 40240321876958496273051651790 2843 (DLS *E*).

35

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$$v = 6, a = 8, \rho = [2, 2, 2, 2, 4, 4].$$

Method of finding:

Brute Force for horizontally symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
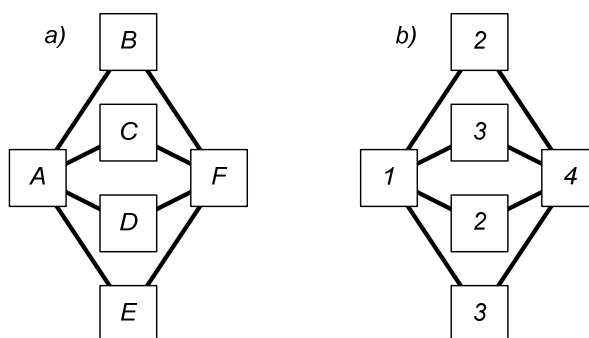
15.3. Rhombus-4 (4 CF)

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204635978371054982625398106475896723014408736219576419
08532935827146084620973516975184203 (DLS *A*, CF 1) − horizontal symmetry;

DLS 2: 01234567899845367120857963041279561238044381902567126074935867982
15043301758429626048719355432098671 (DLS *B*, CF 2);

DLS 3: 01234567899845367120857963041270561238944381902567126074935867082
15943391758420626948710355432098671 (DLS *C*, CF 3);

DLS 4: 01234567899782364510785963024159167834022347908165146052937865948
71023307514289646082159378231097654 (DLS *D*, CF 2);

DLS 5: 01234567899782364510785963024150167834922347908165146052937865048
71923397514280646982150378231097654 (DLS *E*, CF 3);

DLS 6: 01234567891294635078371054982625398106475806723914498736210576410
98532935827146084629073516075184293 (DLS *F*, CF 4) − horizontal symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204635978371054982625398106475896723014408736219957641908
5329358271460846209735169751842 03 (DLS *A*);

CF 2: 01234567891204573968548073912685492603717068341592973681245038956 27
0144972185603635109824726179048 35 (DLS *B*, DLS *D*);

CF 3: 01234567891204365978231590486734697801527891542036568017932497426 38
5104937821605857609324160582174 93 (DLS *C*, DLS *E*);

CF 4: 01234567891204635978293154860746982710359486723150657981024337501 89
4268042367591731509486258679023 14 (DLS *F*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 6$, $a = 8$, $\rho = [2, 2, 2, 2, 4, 4]$.

Method of finding:

Brute Force for horizontally symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

16.1. Symmetric fish

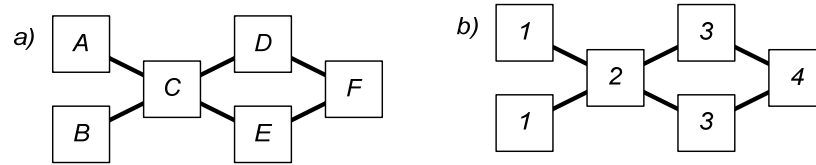
Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789120436795887965240316945182370451073982620578136949371608542386927041554380912677682945103 (DLS *A*, CF 1);

DLS 2: 0123456789473190862524196308577894365012805627349169751842035260819374134209756896875421303508721946 (DLS *C*, CF 2) – horizontal symmetry;

DLS 3: 0123456789120436795887965240316945182370451073982620578136946371908542386927041554380912677982645103 (DLS *D*, CF 3);

DLS 4: 0123456789140236597886957430219267184503371062984550368124977541938260485927031623780916546984507132 (DLS *B*, CF 1);

DLS 5: 0123456789140236597886957430219267184530371062984550368124977541908263485927031623780916546984537102 (DLS *E*, CF 3);

DLS 6: 0123456789473190862574196308522894365017805627349169751842035260819374134709256896825471303508721946 (DLS *F*, CF 4) – horizontal symmetry.

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456789120436795887965240316945182370451073982620578136949371608542386927041554380912677682945103 (DLS *A*, DLS *B*);

CF 2: 0123456789123054967846789012355961278304875936042175961830422805634917934781256060147258933482097156 (DLS *C*);

CF 3: 0123456789120436795887965240316945182370451073982620578136946371908542386927041554380912677982645103 (DLS *D*, DLS *E*);

CF 4: 0123456789120487963589367145022579108346908562147376580432914897365120571093286434625970186341280957 (DLS *F*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 6$, $a = 6$, $\rho = \begin{bmatrix} 1, 1, 2, 2, 2, 4 \end{bmatrix}$.

Method of finding:

Brute Force for horizontally symmetric DLSs in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
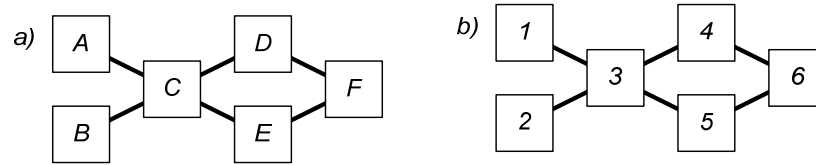
16.2. Asymmetric fish

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789120436795849125780365679143802375861429073912806452037895164958603142784659023716840729513 (DLS *A*, CF 1);

DLS 2: 0123456789945702861352619043784018392567837654190225408791366795213840380916725416327804957984635021 (DLS *C*, CF 2);

DLS 3: 0123456789120536794849125780365679143802374861529073912806542037894165948603152785649023716850729413 (DLS *D*, CF 3);

DLS 4: 0123456789120536794849825710365679143802374168529073982106542037894165941603852785649023716850729413 (DLS *E*, CF 4);

DLS 5: 0123456789120436795849825710365679143802375168429073982106452037895164951603842784659023716840729513 (DLS *B*, CF 5);

DLS 6: 0123456789945702861352691043784018392567837654910225408719366795213840380196725416327804957984635021 (DLS *F*, CF 6).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456789120458396763890475128975612340469713025827368954019860374125501872963434529018767541268093 (DLS *A*);

CF 2: 0123456789120437985658917406232476108395963581740240872359613950681247674209351883695241707518962034 (DLS *B*);

CF 3: 0123456789120436795849125780365679143802375861429073912806452037895164958603142784659023716840729513 (DLS *C*);

CF 4: 0123456789120463795837819602452059783164957210843659368410274697315802731852469084650923716840279513 (DLS *D*);

CF 5: 0123456789120436795849825710365679143802375168429073982106452037895164951603842784659023716840729513 (DLS *E*);

CF 6: 0123456789120458396763890475128975612340269713045847368952019860374125501872963434529018767541268093 (DLS *F*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 6, a = 6, \rho = [1, 1, 2, 2, 2, 4]$.

Method of finding:

search for partially generalized symmetric DLSs with (4,31,31) symmetry, symmetric filled cells number $M = 70$, in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
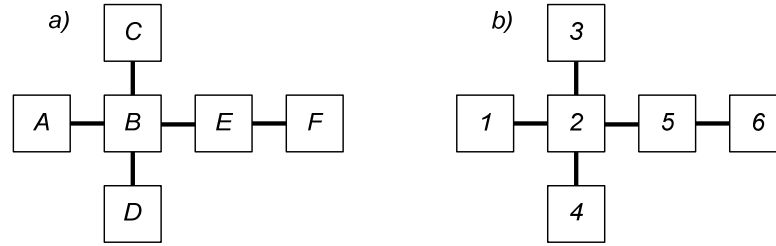
17. Cross

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204678953754201369837682915044079385261938576041268105
29347593684712084519320762697104835 (DLS *A*, CF 1);

DLS 2: 01234567894638597012527916083410958724636842903175871634529029847
31506946021835735076849217351029648 (DLS *B*, CF 2);

DLS 3: 01234567898204671953754208369137612985044079315268938576041268105
29347593614782014589320762697804135 (DLS *C*, CF 3);

DLS 4: 01234567898207641953754208369137612985044079315268938576041268105
29347593617482014589320762694807135 (DLS *E*, CF 4);

DLS 5: 01234567891207648953754201369837682915044079385261938576041268105
29347593687412084519320762694107835 (DLS *D*, CF 5);

DLS 6: 01234567894618597032527936081410958724636842901375873614529029847
13506946023815735076849217351029648 (DLS *F*, CF 6).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204678953754201369837682915044079385261938576041268105 29
34759368471208451932076269710 4835 (DLS *A*);

CF 2: 01234567891230579864796230415827496805316301845927451723809680957 61
34256780924139854123670348691 7205 (DLS *B*);

CF 3: 01234567891206579834371498562089650371426397842501458219036796487 03
21578502614935031624978247931 8056 (DLS *C*);

CF 4: 01234567891204798653465982731065483710929012683547579623480138701 69
42529675401388431905276738501 2964 (DLS *E*);

CF 5: 01234567891204598673965821043734179028652569837014873264950168410 73
95279853641204076125398539078 1246 (DLS *D*);

CF 6: 01234567891234089576358167249026795403184098725631590716482374158 93
06298623071546350218947874693 1205 (DLS *E*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 6,\ a = 5,\ \rho = [1, 1, 1, 1, 2, 4]$.

Method of finding:

search for partially central symmetric DLSs, symmetric filled cells number $M = 60$, in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
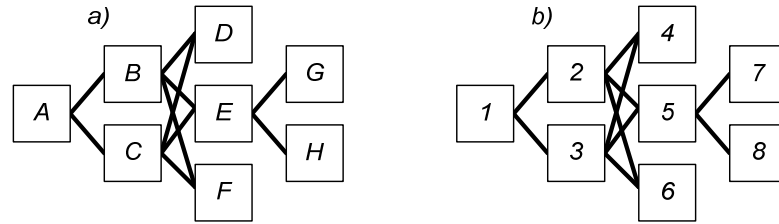
18. Flyer

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204638957409128537669475018237359840162583671249085709 63214978532460136120975482468179035 (DLS *A*, CF 1);

DLS 2: 01234567894387092615165987340292346185706578904231201536984734921 87056786024519387065319245941720368 (DLS *B*, CF 2);

DLS 3: 01234567894837092615165937840292846135706578904231201586934734921 87056736024519887065319245941720863 (DLS *C*, CF 3);

DLS 4: 01234567891204638957409128537669175048237359810462583674219085709 63241978532160436420975182468179035 (DLS *D*, CF 4);

DLS 5: 01234567891204638957409128537669475218037359842160583671049285709 63214978530462136120975482468179035 (DLS *E*, CF 5);

DLS 6: 01234567891204638957409128537669175248037359812460583674019285709 63241978530162436420975182468179035 (DLS *F*, CF 6);

DLS 7: 01234567894387092615165087349292346185706578904231201536984734021 87956786924510387965310245941720368 (DLS *G*, CF 7);

DLS 8: 01234567894837092615165037849292846135706578904231201586934734021 87956736924510887965310245941720863 (DLS *H*, CF 8).

Adjacency matrix:

$$M = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 0123456789120463895740912853766947501823735984016258367124908570963 214978532460136120975482468179035 (DLS *A*);

CF 2: 0123456789120458397626579148039748631520806534719243912780655470169 238793280564135860924176819720354 (DLS *B*);

CF 3: 0123456789120458397626579418039718634520806531749243912780655470169 238793280561435860921476849720351 (DLS *C*);

CF 4: 0123456789120463895740912853766917504823735981046258367421908570963 241978532160436420975182468179035 (DLS *D*);

CF 5: 0123456789120463895740912853766947521803735984216058367104928570963 214978530462136120975482468179035 (DLS *E*);

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<u>CF 6:</u> 01234567891204638957409128537669175248037359812460583674019285709963
24197853016243642097518246817903 5 (DLS *F*);

<u>CF 7:</u> 01234567891206579438346721589076958340219548127603483269017529703485
1680197623546751083942538490126 7 (DLS *G*);

<u>CF 8:</u> 01234567891206579438346721589076859340218549127603483269017529703485
1690187623546751083942539480126 7 (DLS *H*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$$v = 8, a = 10, \rho = [1, 1, 2, 2, 2, 4, 4, 4].$$

Method of finding:
search for partially central symmetric DLSs, symmetric filled cells number $M = 60$, in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
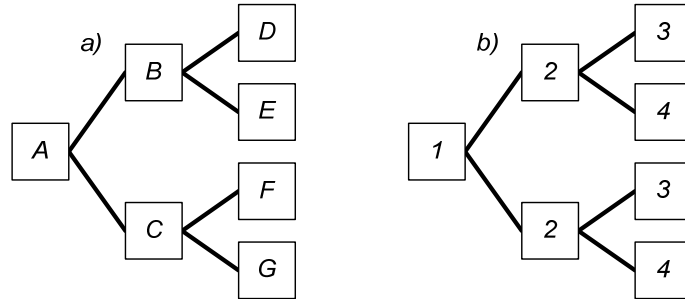
20. Tree-1

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204679853847591063270612385949836521407379208416565478
93210261034597853897620414958107326 (DLS *A*, CF 1) − generalized
symmetry (4,31,31);

DLS 2: 01234567896375892041104962537829081476534582703916583726149072613
89504945607813236905148278714930265 (DLS *B*, CF 2);

DLS 3: 01234567896475892031103962547829081476533582704916584726139072613
89504935607814246905138278714930265 (DLS *C*, CF 2);

DLS 4: 01234567891204679853847591263070612385949836521407379208416565478
93012261034597853897602414958107326 (DLS *D*, CF 3);

DLS 5: 01234567891234679850847591260370612385949806521437379208416565478
93012261034597853897602414958107326 (DLS *E*, CF 4);

DLS 6: 01234567891204679853554789106327061238594983652140737920841656547893
93210261034597883597620414985107326 (DLS *F*, CF 3);

DLS 7: 01234567891208679453587491063270612385949436521807379208416565478
93210261034597883597620414985107326 (DLS *G*, CF 4).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204679853847591063270612385949836521407379208416565478 93
2102610345978538976204149581 07326 (DLS *A*);

CF 2: 01234567891234069857745819063257968432103812975406460728139593815 27
06465493021782075638941 8960714523 (DLS *B*, DLS *C*);

CF 3: 01234567891204587936564893021764591083729360741528879201364548763 25
190391567280425378940617081269453 (DLS *D*, DLS *F*);

CF 4: 01234567891204587936564893021764591083722360741598879201364548763 95
120391567280495378240617081269453 (DLS *E*, DLS *G*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 7, a = 6, \rho = \left[1, 1, 1, 1, 2, 3, 3\right].$$

Method of finding:

> search for generalized symmetric DLSs with (4,31,31) generalized symmetry in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

Remark:

> for different orders N of DLSs it is well known different kinds of trees.
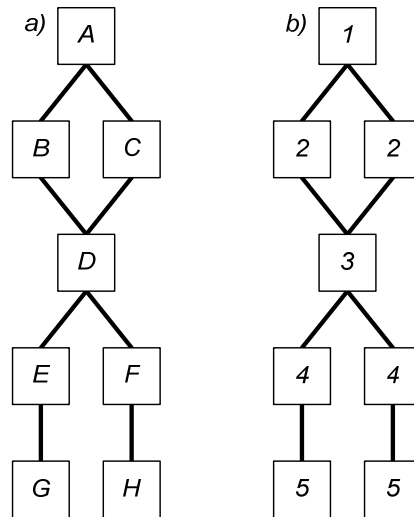
21. Venus

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

<u>DLS 1</u>: 01234567891204789635738560149259168372043652918047907854231645902
73168846719052368413259702739064851 (DLS *A*, CF 1) − generalized symmetry (4,31,31);

<u>DLS 2</u>: 01234567895896217340304856912716720984358760123594425937160824379
05816730168495295847320616915840273 (DLS *B*, CF 2);

<u>DLS 3</u>: 01234567895896217340394856012716729084358769123504425037169824370
95816730168495295847320616015849273 (DLS *C*, CF 2);

<u>DLS 4</u>: 01234567891204769835736580149259182370643856910247907254831645906
73128648719250386413259702739084651 (DLS *D*, CF 3) − generalized symmetry (4,31,31);

<u>DLS 5</u>: 01234567898596217340304856912716720954385760123894428937160524379
08516730168495295847320616915840273 (DLS *E*, CF 4);

<u>DLS 6</u>: 01234567898596217340394856012716729054385769123804428037169524370
98516730168495295847320616015849273 (DLS *F*, CF 4);

<u>DLS 7</u>: 01234567891274069835736580149259182370643856910247970254831645906
73128648719250386413259702039784651 (DLS *G*, CF 5);

<u>DLS 8</u>: 01234567891204769835786530149259182370648356910247907254831645906
73128648719250336418259702739084651 (DLS *H*, CF 5).

Adjacency matrix:

$$
M = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
$$

Different CFs set within combinatorial structure:

<u>CF 1:</u> 01234567891204789635738560149259168372043652918047907854231645902731
688467190523684132597027390648511 (DLS *A*);

<u>CF 2:</u> 01234567891204598637305781942695463278015918702364467018329564952310
78836107495278329651402789640513 (DLS *B*, DLS *C*);

<u>CF 3:</u> 01234567891204769835736580149259182370643856910247907254831645906731
28648719250386413259702739084651 (DLS *D*);

<u>CF 4:</u> 01234567891204598637306781942596453278016918702354457018329654962310
78835107496278329651402789640513 (DLS *E*, DLS *F*);

<u>CF 5:</u> 01234567891204769835786530149259182370648356910247907254831645906731
28648719250336418259702739084651 (DLS *G*, DLS *H*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 8, a = 8, \rho = [1, 1, 2, 2, 2, 2, 2, 4].$$

Method of finding:

search for generalized symmetric DLSs with (4,31,31) generalized symmetry in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

22. Daedalus-8

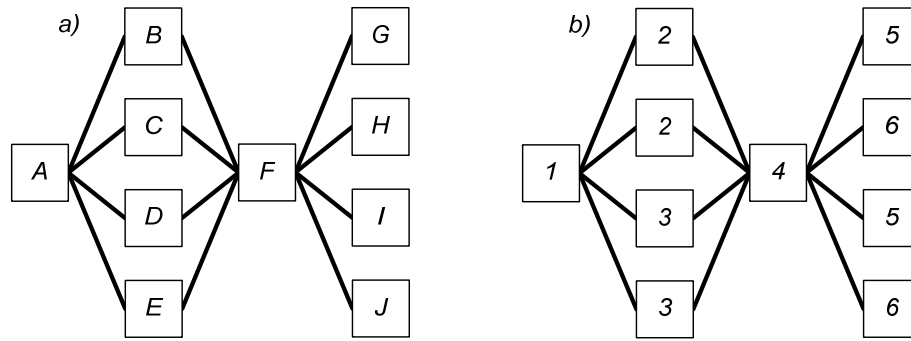Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 0123456789150679342838640251975917384602405283791627915 4086372806
19354637910824596482715308435962071 (DLS *A*, CF 1) − generalized symmetry (4,31,31);

DLS 2: 0123456789534017986264759182307596803124961854207339840 2165780372
64591425138790628697304151702695348 (DLS *B*, CF 2);

DLS 3: 0123456789534017986264759812307596803124968154207339140 2865780372
64591425831790628697304151702695348 (DLS *C*, CF 2);

DLS 4: 0123456789534917086264759182307506893124961854207339840 2165780372
64591425138790628607394151792605348 (DLS *D*, CF 3);

DLS 5: 0123456789534917086264759812307506893124968154207339140 2865780372
64591425831790628607394151792605348 (DLS *E*, CF 3);

DLS 6: 0123456789120679345838640251972917384605405283791657912 4086375806
19324637910854296485712308435962071 (DLS *F*, CF 4) − generalized symmetry (4,31,31);

DLS 7: 0123456789534017986284759612307598603124968154207339140 2865760372
84591425631790828697304151702895346 (DLS *G*, CF 5);

DLS 8: 0123456789534917086284759612307508693124968154207339140 2865760372
84591425631790828607394151792805346 (DLS *H*, CF 6);

DLS 9: 0123456789534017986264759182307596801324963854207119840 2365780172
64593425138790628697304153702695148 (DLS *I*, CF 5);

DLS 10: 0123456789534917086264759182307506891324963854207119840 236578017
26459342513879062860739415379 2605148 (DLS *J*, CF 6).

Adjacency matrix:

$$M = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891239784605956721043853146028972708965314708253914646913780
52847619352068450219733950847261 (DLS *A*);

CF 2: 01234567891204589367859261703496587341202460378951597624081370358916
42431706259868491032753781925406 (DLS *B*, DLS *C*);

CF 3: 01234567891204379568236819745094750821366589731042394281560787316049
25465092387150172683947896540213 (DLS *D*, DLS *E*);

CF 4: 01234567891206793458386402519729173846054052837916579124086375806193
24637910854296485712308435962071 (DLS *F*);

CF 5: 01234567891204589367895261703456987341202460378591957624081370358916
42431706295868491032753781925406 (DLS *G*, DLS *I*);

CF 6: 01234567891204379658245189736056897140329370628145874513092649382615
07356704289160129854737896503214 (DLS *H*, DLS *J*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$$v = 10, \, a = 12, \, \rho = \begin{bmatrix} 1, 1, 1, 1, \, 2, \, 2, \, 2, \, 2, \, 4, \, 8 \end{bmatrix}.$$

Method of finding:

search for generalized symmetric DLSs with (4,31,31) generalized symmetry in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

22. Daedalus-10

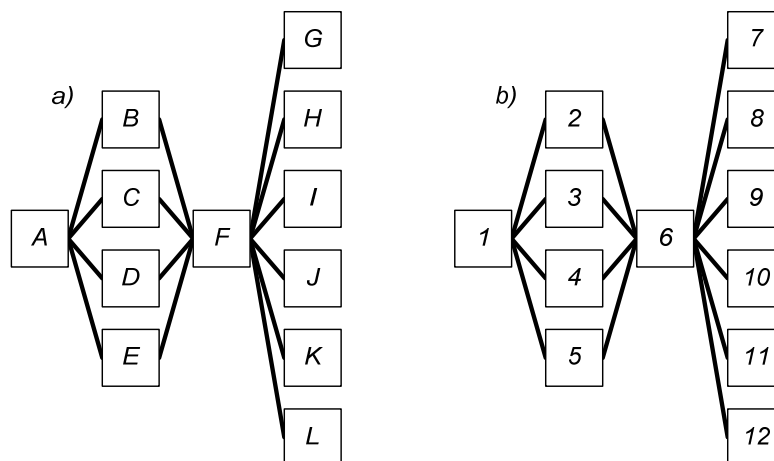
Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204537968946238517045186920373986741205275981034678302
69514867590342163910748525047128693 (DLS *A*, CF 1);

DLS 2: 01234567898756092341639487105292457138602430185976598726410330716
28495486953021715029476387618309524 (DLS *B*, CF 2);

DLS 3: 01234567898756092341639417805292457831602430815976591726480330786
21495486953021715029476387681309524 (DLS *C*, CF 3);

DLS 4: 01234567898756092341239487105696457138206430185972598726410330716
28495486953021715029476387218309564 (DLS *D*, CF 4);

DLS 5: 01234567898756092341239417805696457831206430815972591726480330786
21495486953021715029476387281309564 (DLS *E*, CF 5);

DLS 6: 01234567891204537968976238514075186920343986741205245981037648302
69517867590342163910748525047128693 (DLS *F*, CF 6);

DLS 7: 01234567898756092341639417805292457031682430815976591726480330786
21495486953021715829476307601389524 (DLS *G*, CF 7);

DLS 8: 01234567898756092143639417805292457013682410835976593726480110786
23495486951023735829476107601389524 (DLS *H*, CF 8);

DLS 9: 01234567898756092143639417805292457813602410835976593726480110786
23495486951023735029476187681309524 (DLS *I*, CF 9);

DLS 10: 0123456789875609234123941780569645703128643081597259172648033078
621495486953021715829476307201389564 (DLS *J*, CF 10);

DLS 11: 0123456789875609214323941780569645701328641083597259372648011078
623495486951023735829476107201389564 (DLS *K*, CF 11);

DLS 12: 0123456789875609214323941780569645781320641083597259372648011078
623495486951023735029476187281309564 (DLS *L*, CF 12).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204537968946238517045186920373986741205275981 03467830269 51486759034216391074852 5047128693 (DLS *A*);

CF 2: 01234567891204365978395681740246192708358065723194534218 90672478901 35698375426107590638241 6781094523 (DLS *B*);

CF 3: 01234567891204573968948672510337591684202697304851753824 10968965017 23450718396426340982517 4812690375 (DLS *C*);

CF 4: 01234567891204365978235869140778695301426410729853409281 73653541078 29687359046219687142530 5976283014 (DLS *D*);

CF 5: 01234567891204597638478693015290356428176571389420389276 10457650814 29323681759048419023576 5947208361 (DLS *E*);

CF 6: 01234567891204537968976238514075186920343986741205245981 03764830269 51786759034216391074852 5047128693 (DLS *F*);

CF 7: 01234567891204635978783951460260917285432548903167436718 02955612097 83487503694219486271350 3975842016 (DLS *G*);

CF 8: 01234567891204598637751290346893860152744638721950275918 03463960274 81584713695025847632091 6095847123 (DLS *H*);

CF 9: 01234567891204635978395681740243192708658045723196546218 90372678901 35498375426107590368241 6781094523 (DLS *I*);

CF 10: 01234567891204635978283951460760912785437548903162436718 0295561209 78348750369421948672135 03975842016 (DLS *J*);

CF 11: 01234567891204598637751230946839860152744638721950275918 0346936027 48158471963502584763209 16095847123 (DLS *K*);

CF 12: 01234567891204368957986217534053479816026419520873705863 4291267084 91353985017426879620351 44531792068 (DLS *L*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:
$$v = 12,\, a = 14,\, \rho = \left[1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 4, 10\right].$$

Method of finding:

search for partially central symmetric DLSs, symmetric filled cells number $M = 60$, in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
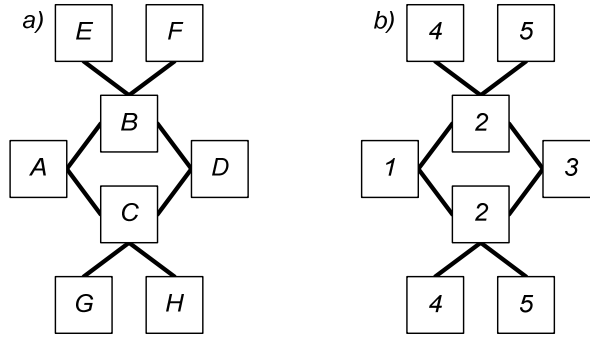
23. Robot

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 012345678912047859632381604597453916207874605198329675038214509834762169428731508716290345385792 1406 (DLS *A*, CF 1) − generalized symmetry (4,31,31);

DLS 2: 012345678983109675244896723015726458193019586423076041375892950721846337851092462679034158543289 0671 (DLS *B*, CF 2);

DLS 3: 012345678983190675244896723015726458193010586423976941375802950721846337851902462670934158543280 9671 (DLS *C*, CF 2);

DLS 4: 012345678912047958632381604597453916207874605189328675039214509834762169428731509716280345385792 1406 (DLS *D*, CF 3) − generalized symmetry (4,31,31);

DLS 5: 012345678912347859602381604597450916237874605198329675038214509834762169428701538716290345385792 1406 (DLS *E*, CF 4);

DLS 6: 012345678912347958602381604597450916237874605189328675039214509834762169428701539716280345385792 1406 (DLS *F*, CF 5);

DLS 7: 012345678952047819632381604597453916207874605198329675038214109834762569428731508756290341381792 5406 (DLS *G*, CF 4);

DLS 8: 012345678952047918632381604597453916207874605189328675039214109834762569428731509756280341381792 5406 (DLS *H*, CF 5).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 012345678912047859632381604597453916207874605198329675038214509834762169428731508716290345385792 1406 (DLS *A*);

CF 2: 012345678912065789349457832601637194052885947132603610289457296830417578490653124785621093503219 7846 (DLS *B*, DLS *C*);

<u>CF 3:</u> 012345678912047958632381604597453916207874605189328675039214509834762169428731509716280345385792140 6 (DLS *D*);

<u>CF 4:</u> 0123456789123457890654926318709756803142308174926563401825978917365024786592041345092176382678094351 (DLS *E*, DLS *G*);

<u>CF 5:</u> 0123456789123459786056108793247346125098685930417285972406139471638205308576294127089134564962081537 (DLS *F*, DLS *H*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 8, a = 8, \rho = [1, 1, 1, 1, 2, 2, 4, 4]$.

Method of finding:

search for generalized symmetric DLSs with (4,31,31) generalized symmetry in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).
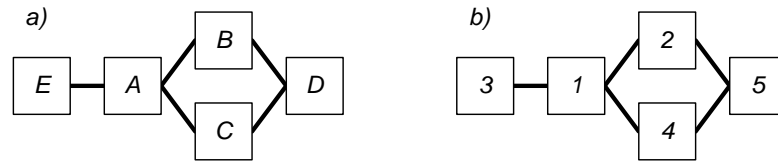
24. Stingray

Fig. Combinatorial structure (a) and corresponding CFs (b)

DLSs within combinatorial structure:

DLS 1: 01234567891204678935583672019437859426102910865347634109752845972
81063765831940294625038718079134256 (DLS *A*, CF 1);

DLS 2: 01234567895492780163637190425826198730459267541830895013247617056
28394358409762140382659177846319502 (DLS *B*, CF 2);

DLS 3: 01234567895410789263637129405826098713453067542891895231047617356
28904958413762042980651377846903512 (DLS *E*, CF 3);

DLS 4: 01234567895412789063637109425826098713453267540891895031247617356
28904958413762040982651377846903512 (DLS *C*, CF 4);

DLS 5: 01234567893204678915581672039417859426302930865147634109752845972
83061765831940294625018738079134256 (DLS *D*, CF 5).

Adjacency matrix:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Different CFs set within combinatorial structure:

CF 1: 01234567891204678935583672019437859426102910865347634109752845972 81
06376583194029462503871807913 4256 (DLS *A*);

CF 2: 01234567891234869057604759281386597301245390178642290568437195823 17
46047160235987468201935387194 5206 (DLS *B*);

CF 3: 01234567891204589637378102459646759312082538670914736089542194523 17
86089462031755019768342689714 2053 (DLS *E*);

CF 4: 01234567891204589367678502149395476128302468370915763089452143719 65
20889562031745019738642389214 7056 (DLS *C*);

CF 5: 01234567891204679853598072361467951402389856317420701986534235672 84
90146380921758472531096234190 8567 (DLS *D*).

Number of vertices, number of edges, ascending sorted vector of vertices powers:

$v = 5, a = 5, \rho = [1, 2, 2, 2, 3]$.

Method of finding:

search for partially central symmetric DLSs, symmetric filled cells number $M = 80$, in combination with method of checking DLS for ODLS (Euler-Parker method, DLX).

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