ON POLYNOMIAL REDUCTION OF PROBLEMS BASED ON DIAGONAL LATIN SQUARES TO THE EXACT COVER PROBLEM. AND RELATED RESULTS...

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Kursk, 2019
**What is Latin squares?**

\[
A = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 9 & 4 & 3 & 6 & 7 & 5 & 0 & 8 \\
2 & 9 & 3 & 1 & 7 & 0 & 5 & 8 & 4 & 6 \\
3 & 4 & 1 & 2 & 8 & 7 & 9 & 6 & 5 & 0 \\
4 & 3 & 5 & 9 & 2 & 1 & 8 & 0 & 6 & 7 \\
5 & 6 & 4 & 8 & 1 & 2 & 0 & 9 & 7 & 3 \\
6 & 5 & 8 & 7 & 0 & 3 & 2 & 1 & 9 & 4 \\
7 & 8 & 6 & 0 & 9 & 4 & 1 & 2 & 3 & 5 \\
8 & 7 & 0 & 5 & 6 & 9 & 3 & 4 & 1 & 2 \\
9 & 0 & 7 & 6 & 5 & 8 & 4 & 3 & 2 & 1 \\
\end{bmatrix}
\]

Normalized LS of order 10

\[
N! \times (N - 1)!
\]

\[
N = |S|
\]

\[
S = \{0, 1, 2, ..., N - 1\}
\]

\[
\begin{align*}
\forall i, j, k = 1, N, j \neq k : (a_{ij} \neq a_{ik}) \land (a_{ji} \neq a_{ki}) \\
\forall i, j = 1, N, i \neq j : (a_{ii} \neq a_{jj}) \land (a_{N-i+1, N-j+1} \neq a_{N-j+1, N-j+1})
\end{align*}
\]

\[
(0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
7 & 2 & 4 & 9 & 0 & 6 & 5 & 1 & 3 & 8 \\
8 & 3 & 6 & 7 & 5 & 9 & 0 & 2 & 4 & 1 \\
2 & 6 & 8 & 5 & 1 & 7 & 4 & 0 & 9 & 3 \\
5 & 8 & 9 & 1 & 7 & 0 & 3 & 4 & 6 & 2 \\
9 & 4 & 1 & 2 & 8 & 3 & 7 & 6 & 0 & 5 \\
4 & 7 & 5 & 6 & 9 & 1 & 8 & 3 & 2 & 0 \\
3 & 0 & 7 & 8 & 2 & 4 & 1 & 9 & 5 & 6 \\
6 & 5 & 0 & 4 & 3 & 2 & 9 & 8 & 1 & 7 \\
1 & 9 & 3 & 0 & 6 & 8 & 2 & 5 & 7 & 4
\end{bmatrix}
\]

Normalized DLS of order 10

\[(N - 1)!\]
Why is this interesting?

Applied problems:
- experiment planning
- cryptography
- error correcting codes
- scheduling
- algebra, combinatorics, statistics, …

Mathematical problems:
- **existence of a triple of MOLS/ MODLS**
- generating functions
- asymptotic behavior of combinatorial characteristics based on DLSs (OEIS)
- number theory (relations between different fields of knowledge)
- magic squares
- Sudoku (LS of order 9 with additional constraints)

Searching for pairs of ODLS of order 10

L. Euler expected that for $N=10$ ODLS doesn’t exist
First pair — Parker et al., 1960

Very rare combinatorial objects:
~30 millions DLS of order 10
has only 1 pair of ODLS!
Closest decision to the triple of MODLS

Orthogonality characteristic
74,
citerra
(world record, 2016)

Orthogonality characteristic
74,
evatutin (2017)

- Can characteristic value be increased? It is open question, we are trying…
- Are decisions differ?
- Are decisions have special properties?
Combinatorial structures for order 10

Strategies:
- direct search (ineffective);
- Euler-Parker method;
- special methods (rows rearrangement, SODLS, etc.).

Main schema:
Generator -> Processor -> Postprocessor

Bottleneck: processor!
New stage (from 04.2019): postprocessor!

Euler-Parker based processor

Two stages:
• getting transversals set;
• getting subsets of N disjoint transversals.

Implementations:
• Brute Force (backtrack programming) — \(\sim 250-300\) DLS/ s;
• Exact cover using (DLX) — \(\sim 600\) (v1) … 900 (v2) DLS/ s;
• Canonizer using (A.D. Belyshev) — \(\sim 6000-8000\) DLS/ s (inside).
Dancing Links X algorithm (abbr. DLX)

Author: Donald E. Knuth, 2000

Problem: Exact Cover Problem (NP class)

https://habr.com/ru/post/194410/
### Exact cover problem example

#### Cover matrix for DLSs of order 3 generation

<table>
<thead>
<tr>
<th>Group 1: $r_i = v$</th>
<th>Group 2: $c_i = v$</th>
<th>Group 3: $a_{ij}$ filled</th>
<th>Groups 4 and 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{00} = 0$</td>
<td>$a_{00} = 1$</td>
<td>$a_{00} = 2$</td>
<td>$a_{01}$</td>
</tr>
<tr>
<td>$a_{01} = 0$</td>
<td>$a_{01} = 1$</td>
<td>$a_{01} = 2$</td>
<td>$a_{02}$</td>
</tr>
<tr>
<td>$a_{02} = 0$</td>
<td>$a_{02} = 1$</td>
<td>$a_{02} = 2$</td>
<td>$a_{10}$</td>
</tr>
<tr>
<td>$a_{10} = 0$</td>
<td>$a_{10} = 1$</td>
<td>$a_{10} = 2$</td>
<td>$a_{11}$</td>
</tr>
<tr>
<td>$a_{11} = 0$</td>
<td>$a_{11} = 1$</td>
<td>$a_{11} = 2$</td>
<td>$a_{12}$</td>
</tr>
<tr>
<td>$a_{12} = 0$</td>
<td>$a_{12} = 1$</td>
<td>$a_{12} = 2$</td>
<td>$a_{20}$</td>
</tr>
<tr>
<td>$a_{20} = 0$</td>
<td>$a_{20} = 1$</td>
<td>$a_{20} = 2$</td>
<td>$a_{21}$</td>
</tr>
<tr>
<td>$a_{21} = 0$</td>
<td>$a_{21} = 1$</td>
<td>$a_{21} = 2$</td>
<td>$a_{22}$</td>
</tr>
</tbody>
</table>

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*Cover matrix for DLSs of order 3 generation*
Latin squares problems -> Exact cover problem

Latin squares connected problems:
• LS/DLS generation;
• special types of LS/DLS generation (normalized, symmetric, etc.);
• getting transversals and diagonal transversals;
• getting disjoint sets of transversals;
• direct search of ODLSs for given DLS.
Dancing links

Fast implementation of cover/uncover operations
Main idea (Brown et al.): find pair of the symmetrically placed transversals, permute rows and columns of LS to get DLS.
Simple transformations
What is intercalate?

Latin square of size 2x2:
- columns $i_1$ and $i_2$;
- rows $j_1$ and $j_2$. 
What is nested intercalates?
**What is loop?**

- full and short loops;
- shortest possible loop is intercalate.
What is Latin subrectangles and Latin subsquares?

- trivial and nontrivial subLRs;
- smallest nontrivial subLR is intercalate.
Strategies of simple transformations

Stage 1 (one of):
• rotate $K$ intercalates;
• rotate $K$ short loops;
• «rotate» $K$ nontrivial subLRs.

Remark: after transformation some squares can be correct LSs, but not DLSs!

Stage 2 (one of):
• Euler-Parker method + DLX (processing for DLSs only, fast);
• Canonizer (processing for LSs, slow, but...).

What strategy is preferable? Experiment:
• average ODLS CF «cost» without postprocessing — 3000 s (8,3 h);
• average ODLS CF «cost» after different simple transformations closure — 4 s ... 16 000 000 s.
Simple transformations: results

- **+10-15%** more ODLS CFs with postprocessing;
- **+72** loop-4 CFs, **+8** line-4 CFs and **+4** 1:3 CFs after postprocessing (very rare combinatorial structures!).
Some results...
Generalized symmetries exploration

Recognition — 2018

Recognition — 2019
Getting ODLS CFs within Gerasim@Home project

Strategy of search: getting source square (random generator, symmetric random generator), try to get orthogonal square, add the unique CF to collection

Recognition — 2018
(1 100 000+ CFs)

Recognition — 2019
(4 800 000+ CFs, 4.3x)
Online Encyclopedia of Integer Sequences

Main classes of DLS:
• A287764 — Number of main classes of diagonal Latin squares of order N (N<9)
• A299783 — Minimal size of main class for diagonal Latin squares of order N with fixed first row (N<9)
• A299784 — Maximal size of main class for diagonal Latin squares of order N with fixed first row (N<9)
• A299785 — Minimal size of main class for diagonal Latin squares of order N (N<9)
• A299787 — Maximal size of main class for diagonal Latin squares of order N (N<9)

Intercalates, loops, Latin subrectangles and subsquares in DLS:
• A307163 — Minimum number of intercalates in a diagonal Latin square of order N (N<9)
• A307164 — Maximum number of intercalates in a diagonal Latin square of order N (N<9)
• A307166 — Minimum number of loops in a diagonal Latin square of order N (N<8)
• A307167 — Maximum number of loops in a diagonal Latin square of order N (N<8)
• A307170 — Minimum number of partial loops in a diagonal Latin square of order N (N<8)
• A307171 — Maximum number of partial loops in a diagonal Latin square of order N (N<8)
• A307839 — Minimum number of Latin subrectangles in a diagonal Latin square of order N (N<8)
• A307840 — Maximum number of Latin subrectangles in a diagonal Latin square of order N (N<8)
• A307841 — Minimum number of nontrivial Latin subrectangles in a diagonal Latin square of order N (N<8)
• A307842 — Maximum number of nontrivial Latin subrectangles in a diagonal Latin square of order N (N<8)

• https://oeis.org
## Online Encyclopedia of Integer Sequences: example of the numerical series A307841

<table>
<thead>
<tr>
<th>A307841</th>
<th>Minimum number of nontrivial Latin subrectangles in a diagonal Latin square of order n.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 0, 0, 12, 0, 51, 0</td>
<td>(list; graph; refs; listen; history; text; internal format)</td>
</tr>
</tbody>
</table>

### Offset
1, 4

### Comments
A Latin subrectangle is an m × k Latin rectangle of a Latin square of order n, 1 ≤ m ≤ n, 1 ≤ k ≤ n.
A nontrivial Latin subrectangle is an m × k Latin rectangle of a Latin square of order n, 1 < m < n, 1 < k < n.

### Crossrefs
Cf. A307839, A307842.
Sequence in context: A307170 A225951 A278711 A257949 A077351 A119530
Adjacent sequences: A207838 A307839 A307840 A207842 A307844 A307845

### Keywords
nonn,more,new

### Author
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### Status
approved

[https://oeis.org](https://oeis.org)
Combinatorial structures of order 1-9

RakeSearch & Gerasim@Home results;

http://evatutin.narod.ru/evatutin_ls_all_structs_n1to8_eng.pdf;
I have some additional minutes? :)

Related works...
Related work

Collecting CFs and new combinatorial structures search:
• triple of MODLS (is it exist?)
• different structures? (including different orders)

GPU implementation of transversal and cover algorithms?

Enumeration problems (OEIS):
• expanding current sequences
• enumerating DLS and ODLS of special kind (row-inverse, symmetric, …) and its CFs

Pseudo triples:
• 3 kinds of pseudo triples, only 1 was investigated in details
Thank you for your attention!

Thanks to all the volunteers who took part in the Gerasim@home project!

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