Southwest State University
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# ABOUT SYMMETRIES IN DIAGONAL LATIN SQUARES

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Tula, Tula State University, 2017

## What is Latin squares?

$$A = \left\| a_{ij} \right\| \qquad \forall i,\ j,k = \overline{1,N},\ j \neq k : \left(a_{ij} \neq a_{ik}\right) \wedge \left(a_{ji} \neq a_{ki}\right)$$

$$i,\ j = \overline{1,N} \qquad \forall i,\ j = \overline{1,N}, i \neq j : \left(a_{ii} \neq a_{jj}\right) \wedge \left(a_{N-i+1,\,N-i+1} \neq a_{N-j+1,\,N-j+1}\right)$$

$$N = \left|S\right|$$

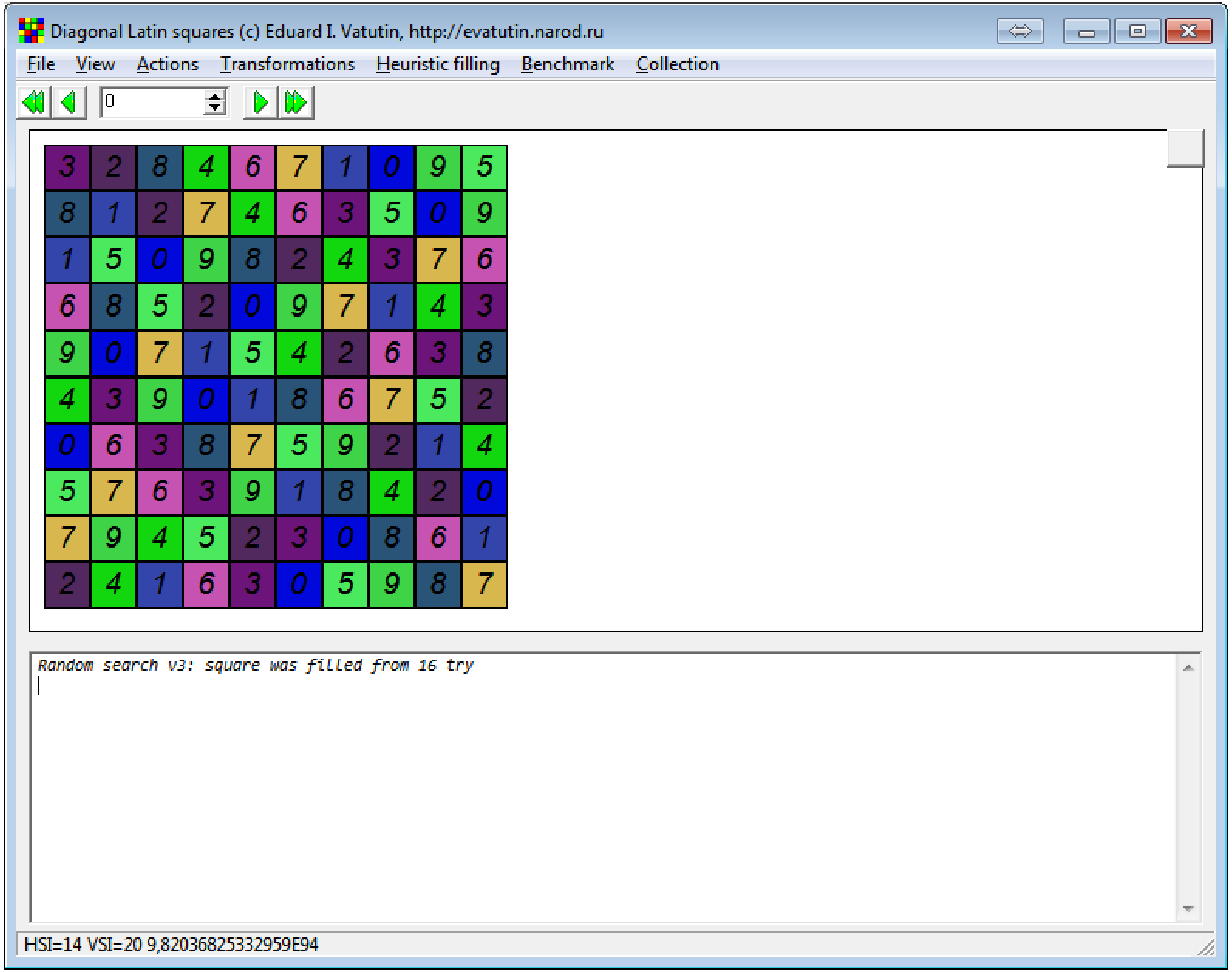
$$S = \left\{0, 1, 2, \ldots, N-1\right\}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 9 & 4 & 3 & 6 & 7 & 5 & 0 & 8 \\ 2 & 9 & 3 & 1 & 7 & 0 & 5 & 8 & 4 & 6 \\ 3 & 4 & 1 & 2 & 8 & 7 & 9 & 6 & 5 & 0 \\ 4 & 3 & 5 & 9 & 2 & 1 & 8 & 0 & 6 & 7 \\ 5 & 6 & 4 & 8 & 1 & 2 & 0 & 9 & 7 & 3 \\ 6 & 5 & 8 & 7 & 0 & 3 & 2 & 1 & 9 & 4 \\ 7 & 8 & 6 & 0 & 9 & 4 & 1 & 2 & 3 & 5 \\ 8 & 7 & 0 & 5 & 6 & 9 & 3 & 4 & 1 & 2 \\ 9 & 0 & 7 & 6 & 5 & 8 & 4 & 3 & 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 4 & 9 & 0 & 6 & 5 & 1 & 3 & 8 \\ 8 & 3 & 6 & 7 & 5 & 9 & 0 & 2 & 4 & 1 \\ 2 & 6 & 8 & 5 & 1 & 7 & 4 & 0 & 9 & 3 \\ 5 & 8 & 9 & 1 & 7 & 0 & 3 & 4 & 6 & 2 \\ 9 & 4 & 1 & 2 & 8 & 3 & 7 & 6 & 0 & 5 \\ 4 & 7 & 5 & 6 & 9 & 1 & 8 & 3 & 2 & 0 \\ 3 & 0 & 7 & 8 & 2 & 4 & 1 & 9 & 5 & 6 \\ 6 & 5 & 0 & 4 & 3 & 2 & 9 & 8 & 1 & 7 \\ 1 & 9 & 3 & 0 & 6 & 8 & 2 & 5 & 7 & 4 \end{pmatrix}$$

Normalized LS of order 10 — Normalized DLS of order 10

$$N! \times (N-1)! \qquad (N-1)!$$

# Lets try to get diagonal Latin square!




- http://evatutin.narod.ru/evatutin_LsEdit.7z

# Why is this interesting?

Applied problems:

- experiment planning
- cryptography
- error correcting codes
- scheduling
- algebra, combinatorics, statistics, ...

Mathematical problems:

- existence of a triple of MOLS/MODLS
- generating functions
- asymptotic behavior of combinatorial characteristics based on DLSs (OEIS)
- number theory (relations between different fields of knowledge)
- magic squares
- Sudoku (LS of order 9 with additional constraints)

# Searching for pairs of ODLS of order 10

L. Euler expected that for N=10 ODLS doesn't exist

First pair — Parker et al., 1960

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 2 | 0 | 4 | 3 | 7 | 9 | 8 | 5 | 6 |
| 7 | 3 | 5 | 9 | 0 | 4 | 8 | 6 | 2 | 1 |
| 3 | 5 | 6 | 8 | 9 | 0 | 4 | 1 | 7 | 2 |
| 4 | 9 | 7 | 2 | 6 | 8 | 1 | 5 | 0 | 3 |
| 5 | 8 | 4 | 6 | 7 | 1 | 3 | 2 | 9 | 0 |
| 8 | 4 | 9 | 1 | 2 | 3 | 7 | 0 | 6 | 5 |
| 6 | 7 | 3 | 0 | 1 | 2 | 5 | 9 | 4 | 8 |
| 9 | 0 | 1 | 5 | 8 | 6 | 2 | 4 | 3 | 7 |
| 2 | 6 | 8 | 7 | 5 | 9 | 0 | 3 | 1 | 4 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 7 | 5 | 1 | 9 | 2 | 8 | 0 | 4 | 6 | 3 |
| 1 | 0 | 3 | 4 | 6 | 7 | 5 | 2 | 9 | 8 |
| 9 | 8 | 4 | 7 | 5 | 2 | 1 | 0 | 3 | 6 |
| 6 | 7 | 9 | 0 | 8 | 3 | 2 | 1 | 5 | 4 |
| 4 | 6 | 5 | 1 | 0 | 9 | 8 | 3 | 2 | 7 |
| 2 | 3 | 8 | 5 | 1 | 6 | 4 | 9 | 7 | 0 |
| 5 | 2 | 7 | 8 | 3 | 4 | 9 | 6 | 0 | 1 |
| 3 | 4 | 6 | 2 | 9 | 0 | 7 | 8 | 1 | 5 |
| 8 | 9 | 0 | 6 | 7 | 1 | 3 | 5 | 4 | 2 |

SAT@Home, 04.2015

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 9 | 0 | 8 | 5 | 6 | 3 | 1 | 2 | 7 |
| 2 | 5 | 7 | 9 | 6 | 4 | 0 | 8 | 1 | 3 |
| 9 | 0 | 4 | 6 | 8 | 7 | 1 | 5 | 3 | 2 |
| 6 | 7 | 5 | 2 | 1 | 3 | 8 | 0 | 9 | 4 |
| 1 | 8 | 3 | 5 | 7 | 2 | 9 | 6 | 4 | 0 |
| 7 | 3 | 1 | 0 | 9 | 8 | 4 | 2 | 6 | 5 |
| 8 | 2 | 6 | 4 | 0 | 9 | 5 | 3 | 7 | 1 |
| 3 | 4 | 8 | 1 | 2 | 0 | 7 | 9 | 5 | 6 |
| 5 | 6 | 9 | 7 | 3 | 1 | 2 | 4 | 0 | 8 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 6 | 5 | 9 | 7 | 0 | 8 | 2 | 3 | 1 | 4 |
| 4 | 7 | 1 | 2 | 3 | 9 | 8 | 0 | 6 | 5 |
| 1 | 2 | 0 | 4 | 5 | 3 | 7 | 6 | 9 | 8 |
| 2 | 6 | 8 | 0 | 9 | 4 | 1 | 5 | 3 | 7 |
| 8 | 4 | 6 | 9 | 2 | 7 | 0 | 1 | 5 | 3 |
| 5 | 0 | 4 | 6 | 8 | 2 | 3 | 9 | 7 | 1 |
| 9 | 3 | 5 | 1 | 7 | 6 | 4 | 8 | 0 | 2 |
| 7 | 8 | 3 | 5 | 6 | 1 | 9 | 4 | 2 | 0 |
| 3 | 9 | 7 | 8 | 1 | 0 | 5 | 2 | 4 | 6 |

Gerasim@Home, 04.2017

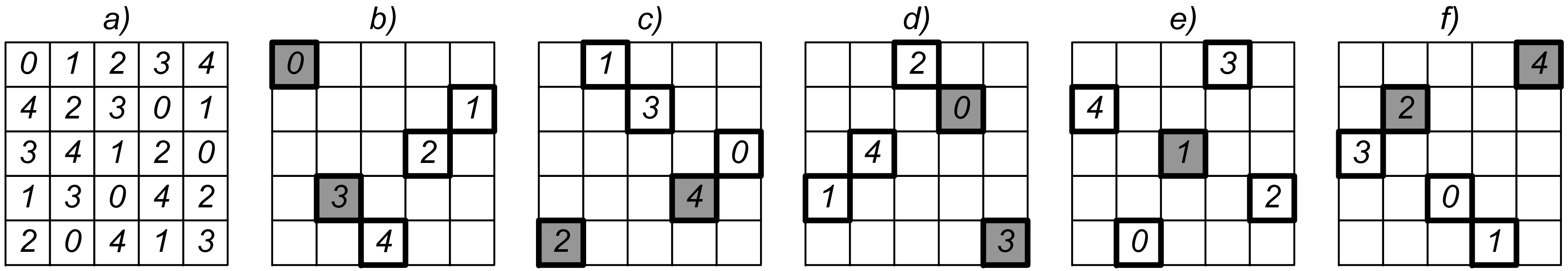
Present for citerra, 2017 :)

Very rare combinatorial objects:
**~30 millions DLS** of order 10
has only **1 pair of ODLS**!

We need to use transversals...

# Searching for ODLS: approaches

- Brute Force + backtracking + clippings + ordering + ... (very long)
- SAT (some tens of hours, long)
- filling by pairs of elements [$a_{ij}$, $b_{ij}$] (long)
- using transversals (fast) – **<u>200 – 800 DLS/s</u>** for different algorithms!




$T^{(d)}_1=\{a_{11}, a_{25}, a_{34}, a_{42}, a_{53}\}$

$T^{(d)}_2=\{a_{12}, a_{23}, a_{35}, a_{43}, a_{51}\}$

$T^{(d)}_3=\{a_{13}, a_{24}, a_{32}, a_{41}, a_{55}\}$

$T^{(d)}_4=\{a_{14}, a_{21}, a_{33}, a_{45}, a_{52}\}$

$T^{(d)}_5=\{a_{15}, a_{22}, a_{31}, a_{43}, a_{54}\}$

## Crossing and orthogonal transversals

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 8 | 4 | 6 | 7 | 1 | 0 | 9 | 5 |
| 8 | 1 | 2 | 7 | 4 | 6 | 3 | 5 | 0 | 9 |
| 1 | 5 | 0 | 9 | 8 | 2 | 4 | 3 | 7 | 6 |
| 6 | 8 | 5 | 2 | 0 | 9 | 7 | 1 | 4 | 3 |
| 9 | 0 | 7 | 1 | 5 | 4 | 2 | 6 | 3 | 8 |
| 4 | 3 | 9 | 0 | 1 | 8 | 6 | 7 | 5 | 2 |
| 0 | 6 | 3 | 8 | 7 | 5 | 9 | 2 | 1 | 4 |
| 5 | 7 | 6 | 3 | 9 | 1 | 8 | 4 | 2 | 0 |
| 7 | 9 | 4 | 5 | 2 | 3 | 0 | 8 | 6 | 1 |
| 2 | 4 | 1 | 6 | 3 | 0 | 5 | 9 | 8 | 7 |

$$T_1 \cap T_2 = \{3, 7, 5, 1\}$$

$$T_1 \perp T_3 \; (T_1 \cap T_2 = \varnothing)$$

$$T_2 \cap T_3 = \{6\}$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 3 | | | | | | | | | |
| | | | 7 | | | | | | |
| | | | | | 2 | | | | |
| | | | | 0 | | | | | |
| | | | | | | | 6 | | |
| | | | | | | | | 5 | |
| | | | | | | | | | 4 |
| | | | | | | 8 | | | |
| | 9 | | | | | | | | |
| | | 1 | | | | | | | |

Transversal 1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 3 | | | | | | | | | |
| | | | 7 | | | | | | |
| | | | | | | | | | 6 |
| | 8 | | | | | | | | |
| | | | | | 4 | | | | |
| | | | | | | | | 5 | |
| | | | | | | | 2 | | |
| | | | | 9 | | | | | |
| | | | | | | 0 | | | |
| | | 1 | | | | | | | |

Transversal 2

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | | | | | | | | |
| | | | | | | 3 | | | |
| | | | | | | | | | 6 |
| | | | | | 9 | | | | |
| | | 7 | | | | | | | |
| | | | | 1 | | | | | |
| 0 | | | | | | | | | |
| | | | | | | | 4 | | |
| | | | 5 | | | | | | |
| | | | | | | | | 8 | |

Transversal 3

## Some combinatorial characteristics of DLS

THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Minimal and maximal number of transversals:

**1, 0, 0, 8, 3, 32, 7, <u>8</u>** (N < 9), evatutin, veinamond, 2017

**1, 0, 0, 8, 15, 32, 133, <u>384</u>** (N < 9), evatutin, veinamond, 2017

Minimal and maximal number of *diagonal* transversals:

**1, 0, 0, 4, 1, 2, 0, <u>0</u>** (N < 9), evatutin, veinamond, 2017

**1, 0, 0, 4, 5, 6, 27, <u>120</u>** (N < 9), evatutin, veinamond, 2017

**<u>Bolded red values</u>** calculated using Gerasim@Home project

(1 week with 1,5 TFLOP/s real performance)

http://gerasim.boinc.ru

- analytic calculating of presented sequences is [impossible?/difficult] (do you know formulas for them?)
- sequences was reviewed and added to OEIS by our collective!

## Searching for triples of MODLS of order 10: are they exists?

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 2 | 3 | 4 | 9 | 0 | 5 | 6 | 7 | 8 |
| 4 | 0 | 8 | 7 | 6 | 3 | 2 | 1 | 9 | 5 |
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| 5 | 9 | 1 | 2 | 3 | 6 | 7 | 8 | 0 | 4 |
| 3 | 5 | 9 | 8 | 2 | 7 | 1 | 0 | 4 | 6 |
| 2 | 3 | 4 | 0 | 8 | 1 | 9 | 5 | 6 | 7 |
| 7 | 6 | 5 | 9 | 1 | 8 | 0 | 4 | 3 | 2 |
| 6 | 4 | 0 | 1 | 7 | 2 | 8 | 9 | 5 | 3 |
| 8 | 7 | 6 | 5 | 0 | 9 | 4 | 3 | 2 | 1 |

Orthogonality characteristic
74,
citerra
(**world record, 2016**)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| 5 | 0 | 6 | 8 | 7 | 2 | 1 | 3 | 9 | 4 |
| 1 | 6 | 4 | 7 | 9 | 0 | 2 | 5 | 3 | 8 |
| 4 | 9 | 3 | 1 | 2 | 7 | 8 | 6 | 0 | 5 |
| 8 | 3 | 5 | 2 | 0 | 9 | 7 | 4 | 6 | 1 |
| 3 | 7 | 0 | 4 | 8 | 1 | 5 | 9 | 2 | 6 |
| 7 | 4 | 8 | 9 | 6 | 3 | 0 | 1 | 5 | 2 |
| 2 | 5 | 1 | 0 | 3 | 6 | 9 | 8 | 4 | 7 |
| 6 | 2 | 9 | 5 | 1 | 8 | 4 | 0 | 7 | 3 |

Orthogonality characteristic
74,
evatutin (2017)

- Can characteristic value be increased? It is open question...
- Are decisions differ?
- Are decisions have special properties?

# Special types of squares and its properties

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 4 | 2 | 0 | 5 | 3 | 1 |
| 5 | 4 | 3 | 2 | 1 | 0 |
| 2 | 5 | 4 | 1 | 0 | 3 |
| 3 | 0 | 1 | 4 | 5 | 2 |
| 1 | 3 | 5 | 0 | 2 | 4 |

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 4 | 2 | 5 | 0 | 3 | 1 |
| 3 | 5 | 1 | 2 | 0 | 4 |
| 5 | 3 | 0 | 4 | 1 | 2 |
| 2 | 4 | 3 | 1 | 5 | 0 |
| 1 | 0 | 4 | 5 | 2 | 3 |

Symmetric DLS examples

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| 8 | 0 | 6 | 7 | 9 | 3 | 4 | 5 | 2 | 1 |
| 1 | 2 | 5 | 4 | 3 | 9 | 7 | 6 | 0 | 8 |
| 7 | 9 | 3 | 1 | 2 | 6 | 8 | 0 | 4 | 5 |
| 6 | 5 | 1 | 9 | 0 | 7 | 2 | 8 | 3 | 4 |
| 5 | 4 | 0 | 8 | 6 | 2 | 1 | 3 | 9 | 7 |
| 3 | 6 | 4 | 5 | 1 | 8 | 0 | 9 | 7 | 2 |
| 4 | 3 | 8 | 2 | 7 | 0 | 9 | 1 | 5 | 6 |
| 2 | 7 | 9 | 0 | 8 | 1 | 5 | 4 | 6 | 3 |

String-inverse DLS

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 9 | 6 | 4 | 8 | 1 | 3 | 0 | 2 | 7 |
| 9 | 0 | 1 | 8 | 6 | 2 | 7 | 4 | 5 | 3 |
| 4 | 6 | 5 | 2 | 0 | 7 | 8 | 3 | 9 | 1 |
| 2 | 4 | 9 | 7 | 3 | 6 | 1 | 8 | 0 | 5 |
| 3 | 7 | 8 | 9 | 5 | 4 | 0 | 2 | 1 | 6 |
| 7 | 8 | 3 | 0 | 2 | 9 | 5 | 1 | 6 | 4 |
| 8 | 5 | 7 | 1 | 9 | 0 | 4 | 6 | 3 | 2 |
| 6 | 3 | 4 | 5 | 1 | 8 | 2 | 9 | 7 | 0 |
| 1 | 2 | 0 | 6 | 7 | 3 | 9 | 5 | 4 | 8 |

SODLS

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 6 | 2 | 3 | 7 | 0 | 4 | 5 | 1 |
| 4 | 5 | 1 | 0 | 7 | 6 | 2 | 3 |
| 5 | 6 | 7 | 4 | 3 | 0 | 1 | 2 |
| 7 | 3 | 6 | 2 | 5 | 1 | 4 | 0 |
| 2 | 7 | 4 | 1 | 6 | 3 | 0 | 5 |
| 3 | 0 | 5 | 6 | 1 | 2 | 7 | 4 |
| 1 | 4 | 0 | 5 | 2 | 7 | 3 | 6 |

Double symmetric DLS

# Related combinatorial characteristics of DLS

DLS main classes amount:

**1, 0, 0, 1, 2, 2, 972, 4 873 096** (N < 10), evatutin, whitefox, 2017

http://forum.boinc.ru/default.aspx?g=posts&m=87549#post87549

Number of the normalized symmetric and double symmetric DLS:

**0, 2, 64, 3 612 672, 82 731 715 264 512** (N < 11), evatutin, 2017

**0, 2, 0, ~~15 780~~ 12 288** (N < 9), evatutin, 2017

Value **15 780** was wrong due to incomplete symmetry definition (thanks to Alexey D. Belyshev (whitefox) for comments!)

Number of reduced pairs of orthogonal diagonal Latin squares:

**1, 0, 0, 2, 4, 0, 320** (N < 8), evatutin, 2017

Maximum number of orthogonal diagonal Latin squares for one diagonal Latin square:

**1, 0, 0, 1, 1, 0, 3, 824, ≥516, ≥8** (N < 11), hoarfrost, evatutin, 2017

Sequences was also reviewed and added to OEIS by our collective!

## Some else symmetries?

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 6 | 3 | 0 | 2 | 7 | 8 | 1 | 4 | 5 |
| 3 | 2 | 1 | 8 | 6 | 7 | 0 | 5 | 4 |
| 7 | 8 | 6 | 5 | 1 | 3 | 4 | 0 | 2 |
| 8 | 6 | 4 | 7 | 2 | 0 | 5 | 3 | 1 |
| 2 | 7 | 5 | 6 | 8 | 4 | 3 | 1 | 0 |
| 5 | 4 | 7 | 0 | 3 | 1 | 8 | 2 | 6 |
| 4 | 5 | 8 | 1 | 0 | 2 | 7 | 6 | 3 |
| 1 | 0 | 3 | 4 | 5 | 6 | 2 | 8 | 7 |

Centrally symmetric DLS examples

Number of centrally symmetric diagonal Latin squares of order n with constant first row:

**1, 0, 0, 2, 8, 0, 2816, 135 168, 327 254 016** (N < 10), evatutin, 2017

Exist only for N != 4n+2, doesn't exist for N=10 :(

## Formulas for symmetries

[i,j] <=> [i',j']=f([i,j])

[i,j] <=> [i, N – 1 – j] — horizontal symmetry
[i,j]<=>[N – 1 – i, j] — vertical symmetry
[i,j]<=>[N – 1 – i, N – 1 – j] — central symmetry

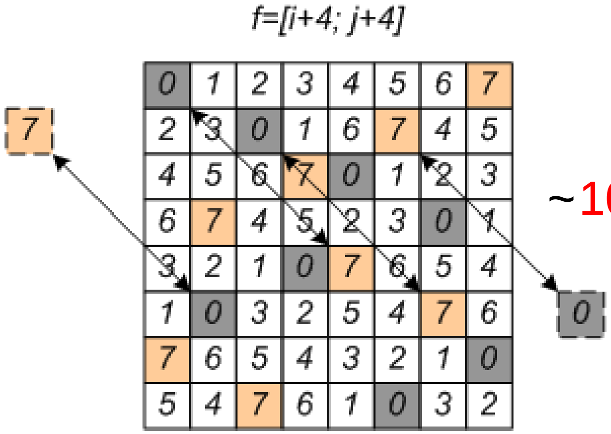
Simple way: using formulas like f(i, j) = Ai + Bj + C and g(i, j) = Di + Ej + F

Tuple **(A, B, C, D, E, F)** identifies the symmetry!

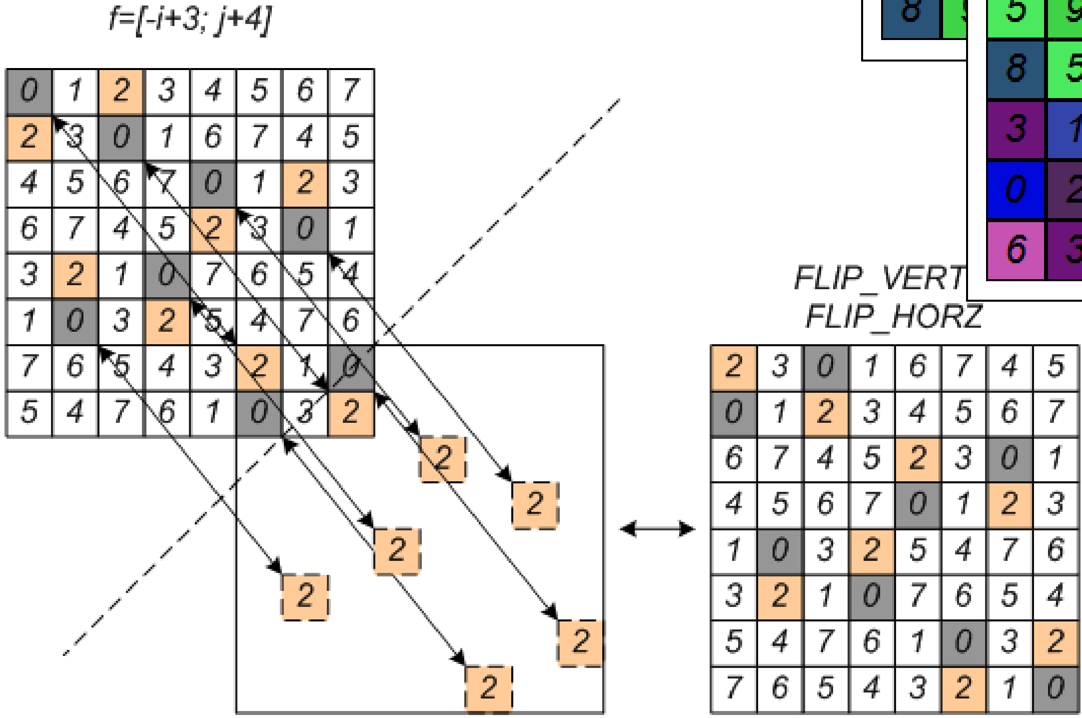
**(1, 0, 0, 0, –1, N – 1)** — horizontal symmetry
**(–1 , 0, N – 1, 0, 1, 0)** — vertical symmetry
**(–1, 0, N – 1, 0, –1, N–1)** — central symmetry

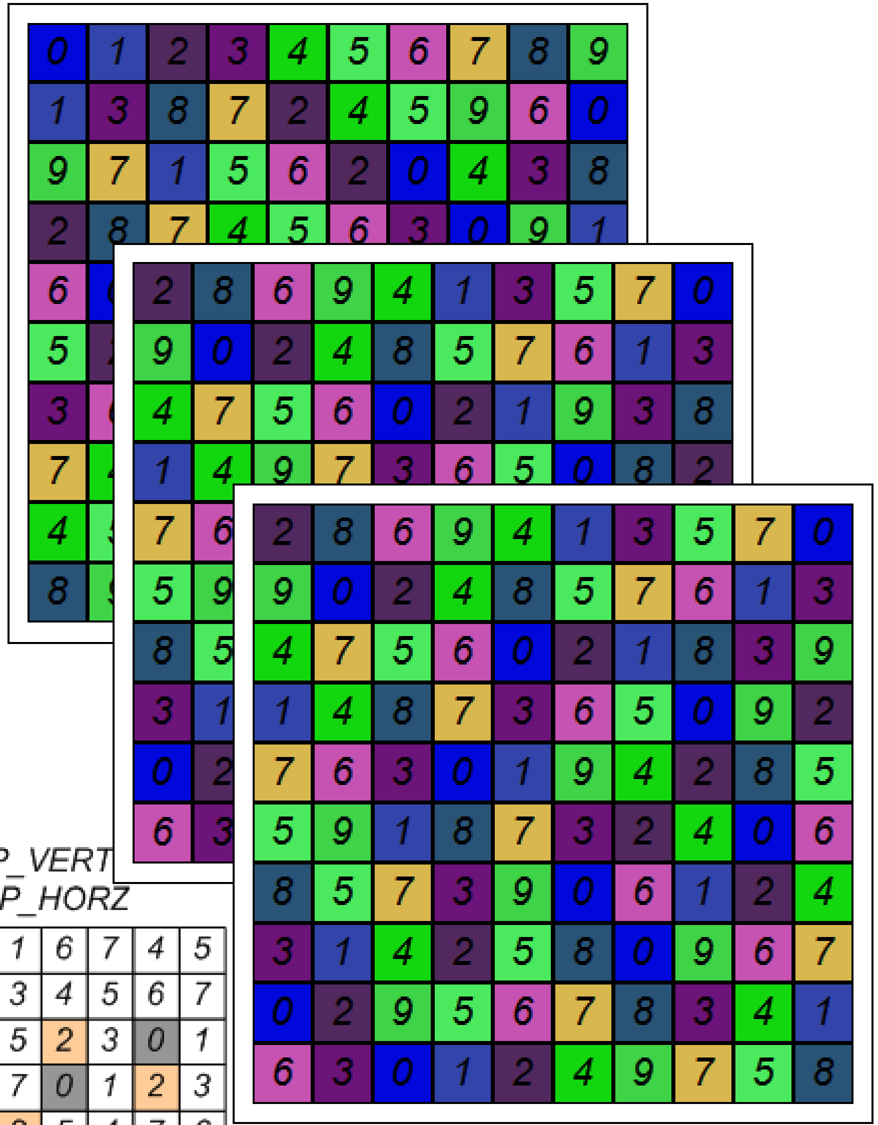
...and at least 13 different (generalized) symmetries for DLS of order 10!!!

# Generalized symmetries example




very very rare:
~1000 **1:1** vs 1 **2:1**!



**2:1 structure**

# Torus movements

Square 1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 3 | 8 | 7 | 2 | 4 | 5 | 9 | 6 | 0 |
| 9 | 7 | 1 | 5 | 6 | 2 | 0 | 4 | 3 | 8 |
| 2 | 8 | 7 | 4 | 5 | 6 | 3 | 0 | 9 | 1 |
| 6 | 0 | 4 | 8 | 9 | 7 | 1 | 3 | 5 | 2 |
| 5 | 2 | 3 | 0 | 1 | 8 | 9 | 6 | 7 | 4 |
| 3 | 6 | 9 | 2 | 0 | 1 | 7 | 8 | 4 | 5 |
| 7 | 4 | 0 | 6 | 8 | 9 | 2 | 5 | 1 | 3 |
| 4 | 5 | 6 | 9 | 3 | 0 | 8 | 1 | 2 | 7 |
| 8 | 9 | 5 | 1 | 7 | 3 | 4 | 2 | 0 | 6 |

FLIP_HORZ

Square 2

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| 0 | 6 | 9 | 5 | 4 | 2 | 7 | 8 | 3 | 1 |
| 8 | 3 | 4 | 0 | 2 | 6 | 5 | 1 | 7 | 9 |
| 1 | 9 | 0 | 3 | 6 | 5 | 4 | 7 | 8 | 2 |
| 2 | 5 | 3 | 1 | 7 | 9 | 8 | 4 | 0 | 6 |
| 4 | 7 | 6 | 9 | 8 | 1 | 0 | 3 | 2 | 5 |
| 5 | 4 | 8 | 7 | 1 | 0 | 2 | 9 | 6 | 3 |
| 3 | 1 | 5 | 2 | 9 | 8 | 6 | 0 | 4 | 7 |
| 7 | 2 | 1 | 8 | 0 | 3 | 9 | 6 | 5 | 4 |
| 6 | 0 | 2 | 4 | 3 | 7 | 1 | 5 | 9 | 8 |

THORUS_UP(5)

Square 3

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 7 | 6 | 9 | 8 | 1 | 0 | 3 | 2 | 5 |
| 5 | 4 | 8 | 7 | 1 | 0 | 2 | 9 | 6 | 3 |
| 3 | 1 | 5 | 2 | 9 | 8 | 6 | 0 | 4 | 7 |
| 7 | 2 | 1 | 8 | 0 | 3 | 9 | 6 | 5 | 4 |
| 6 | 0 | 2 | 4 | 3 | 7 | 1 | 5 | 9 | 8 |
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| 0 | 6 | 9 | 5 | 4 | 2 | 7 | 8 | 3 | 1 |
| 8 | 3 | 4 | 0 | 2 | 6 | 5 | 1 | 7 | 9 |
| 1 | 9 | 0 | 3 | 6 | 5 | 4 | 7 | 8 | 2 |
| 2 | 5 | 3 | 1 | 7 | 9 | 8 | 4 | 0 | 6 |

COPY

Square 3

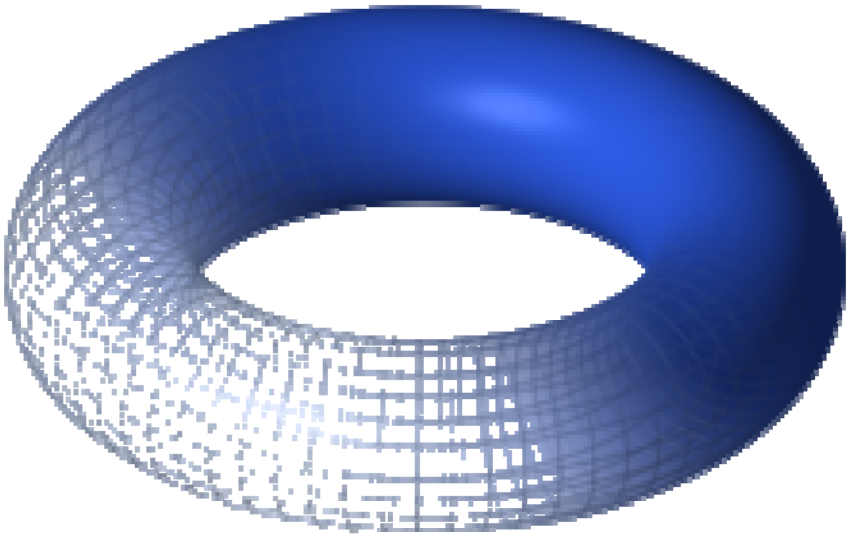
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 7 | 6 | 9 | 8 | 1 | 0 | 3 | 2 | 5 |
| 5 | 4 | 8 | 7 | 1 | 0 | 2 | 9 | 6 | 3 |
| 3 | 1 | 5 | 2 | 9 | 8 | 6 | 0 | 4 | 7 |
| 7 | 2 | 1 | 8 | 0 | 3 | 9 | 6 | 5 | 4 |
| 6 | 0 | 2 | 4 | 3 | 7 | 1 | 5 | 9 | 8 |
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| 0 | 6 | 9 | 5 | 4 | 2 | 7 | 8 | 3 | 1 |
| 8 | 3 | 4 | 0 | 2 | 6 | 5 | 1 | 7 | 9 |
| 1 | 9 | 0 | 3 | 6 | 5 | 4 | 7 | 8 | 2 |
| 2 | 5 | 3 | 1 | 7 | 9 | 8 | 4 | 0 | 6 |

THORUS_LEFT(5)

SYMMETRY

Square 4

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 |
| 0 | 2 | 9 | 6 | 3 | 5 | 4 | 8 | 7 | 1 |
| 8 | 6 | 0 | 4 | 7 | 3 | 1 | 5 | 2 | 9 |
| 3 | 9 | 6 | 5 | 4 | 7 | 2 | 1 | 8 | 0 |
| 7 | 1 | 5 | 9 | 8 | 6 | 0 | 2 | 4 | 3 |
| 4 | 3 | 2 | 1 | 0 | 9 | 8 | 7 | 6 | 5 |
| 2 | 7 | 8 | 3 | 1 | 0 | 6 | 9 | 5 | 4 |
| 6 | 5 | 1 | 7 | 9 | 8 | 3 | 4 | 0 | 2 |
| 5 | 4 | 7 | 8 | 2 | 1 | 9 | 0 | 3 | 6 |
| 9 | 8 | 4 | 0 | 6 | 2 | 5 | 3 | 1 | 7 |



## Additional combinatorial structures

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 2 | 3 | 0 | 6 | 9 | 7 | 8 | 5 | 4 |
| 4 | 3 | 9 | 8 | 0 | 7 | 5 | 1 | 6 | 2 |
| 5 | 7 | 4 | 1 | 9 | 3 | 2 | 6 | 0 | 8 |
| 8 | 9 | 6 | 7 | 5 | 0 | 3 | 2 | 4 | 1 |
| 2 | 8 | 5 | 4 | 7 | 6 | 0 | 9 | 1 | 3 |
| 7 | 5 | 1 | 6 | 2 | 4 | 8 | 3 | 9 | 0 |
| 6 | 0 | 8 | 2 | 3 | 1 | 9 | 4 | 7 | 5 |
| 9 | 4 | 7 | 5 | 8 | 2 | 1 | 0 | 3 | 6 |
| 3 | 6 | 0 | 9 | 1 | 8 | 4 | 5 | 2 | 7 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 2 | 3 | 4 | 5 | 7 | 0 | 9 | 6 | 8 |
| 4 | 7 | 8 | 6 | 9 | 2 | 3 | 5 | 1 | 0 |
| 8 | 9 | 6 | 7 | 2 | 0 | 4 | 3 | 5 | 1 |
| 3 | 0 | 5 | 9 | 6 | 8 | 1 | 4 | 2 | 7 |
| 9 | 5 | 4 | 8 | 1 | 3 | 7 | 2 | 0 | 6 |
| 6 | 4 | 9 | 0 | 7 | 1 | 5 | 8 | 3 | 2 |
| 5 | 8 | 7 | 2 | 0 | 6 | 9 | 1 | 4 | 3 |
| 7 | 3 | 0 | 1 | 8 | 4 | 2 | 6 | 9 | 5 |
| 2 | 6 | 1 | 5 | 3 | 9 | 8 | 0 | 7 | 4 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 2 | 3 | 4 | 7 | 6 | 8 | 9 | 5 | 0 |
| 7 | 4 | 5 | 6 | 2 | 9 | 1 | 3 | 0 | 8 |
| 2 | 8 | 1 | 7 | 0 | 3 | 4 | 5 | 9 | 6 |
| 8 | 9 | 4 | 1 | 3 | 2 | 7 | 0 | 6 | 5 |
| 9 | 0 | 6 | 2 | 1 | 8 | 5 | 4 | 3 | 7 |
| 3 | 6 | 7 | 0 | 5 | 1 | 9 | 8 | 2 | 4 |
| 4 | 3 | 8 | 5 | 9 | 7 | 0 | 6 | 1 | 2 |
| 5 | 7 | 9 | 8 | 6 | 0 | 2 | 1 | 4 | 3 |
| 6 | 5 | 0 | 9 | 8 | 4 | 3 | 2 | 7 | 1 |

**4:1 structures (very very very rare :), 50x vs 2:1)**

2:1 pace of obtaining solutions:

- **0,5 – 1 per day** within Gerasim@Home project (~600 PCs, ~5 TFLOP/s)
- **20 – 30 per day** on Core i7 4770 (Haswell) with 4 threads

## Search and collecting of the unique ODLS CFs

On 19.11.2017 collection includes **427 614** unique ODLS CFs (1 CF — isomorphism class of 7 680 or 15 360 DLS). Available for free access at:

http://forum.boinc.ru/default.aspx?g=posts&m=88700
http://forum.boinc.ru/resource.ashx?a=2940

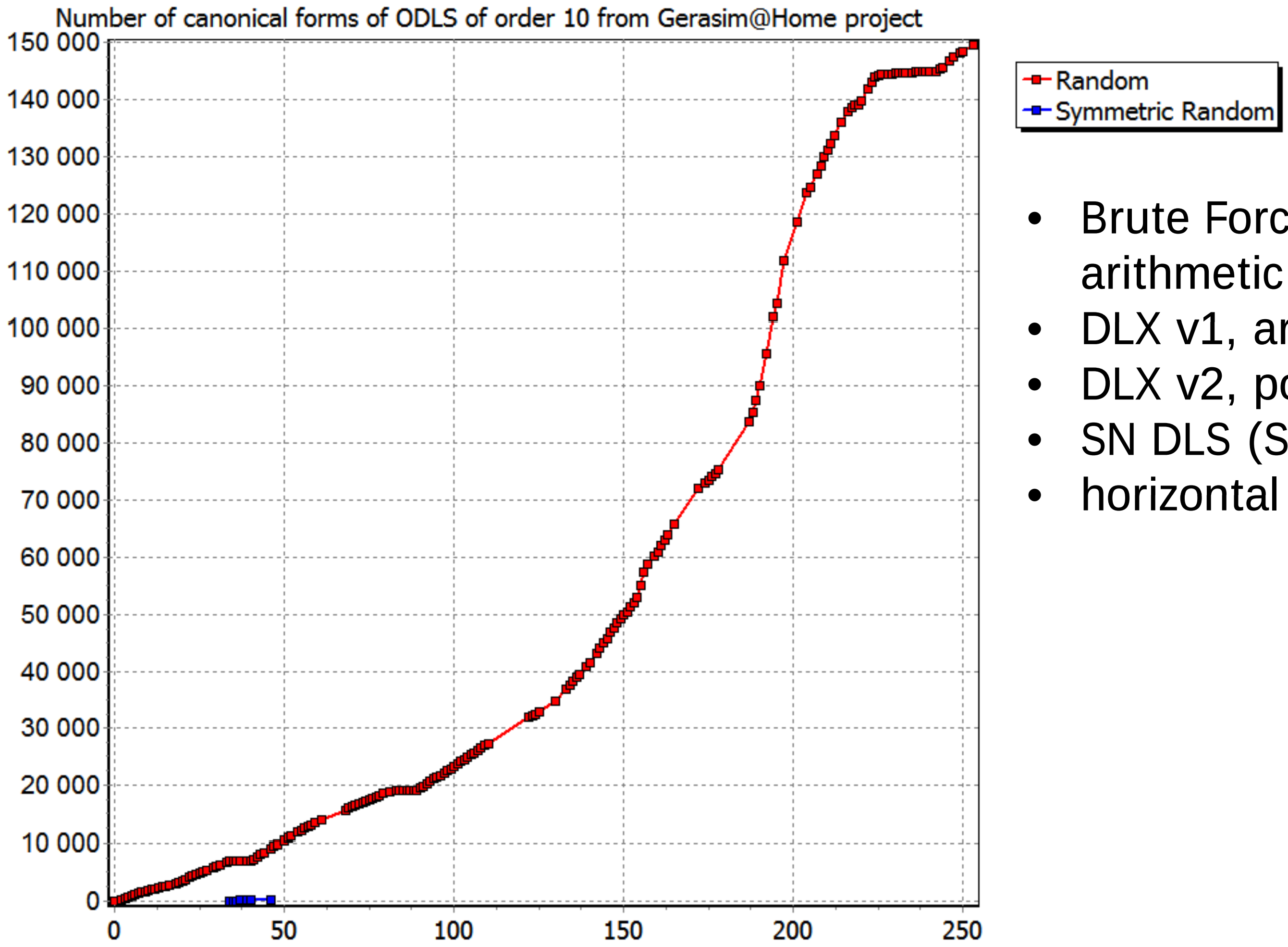
From collection items:

- **6** CFs (Parker, Brown, 1960 – 2000)
- **144** CFs (Nauchnik + SAT@Home volunteers, 2013 – 2015)
- **>2 739** symmetric CFs (citerra, evatutin + Gerasim@Home volunteers, from 2016)
- **1 227** «Brown» CFs (Brown, whitefox, citerra, 2016 – 2017)
- **30 502** SODLS CFs (H. White + whitefox, 2017)
- **149 570** CFs from Gerasim@Home project (evatutin + Gerasim@Home volunteers, 2017), **~ 1 000 CFs per day**
- **>137 931** CFs from ODLK@Home project (Progger + ODLK@Home volunteers, 2017)
- **~300** CFs (different sources, 2000)

Triple of MODLS or pseudo triple with orthogonality characteristic greater than 74 not found!

## Getting ODLS CFs within Gerasim@Home project

Strategy of search: getting source square (random generator, symmetric random generator), try to get orthogonal square, add the unique CF to collection




- Brute Force with bits arithmetic (03.2017)
- DLX v1, array (04.2017)
- DLX v2, pointers (05.2017)
- SN DLS (SCFs) (08.2017)
- horizontal symmetry (10.2017)

## I have some additional minutes? :)

Related works...

## Enumerating the number of DLS

Approaches:

- Brute Force, backtracking, SAT — **0,01 DLS/s** (2014)
- diagonal filling — **28 DLS/s**
- out of order filling cells with |S|=1 — **15 000 DLS/s**
- fast check for sets of available values — **38 000 DLS/s**
- early clipping for cells with |S|=0 — **101 000 DLS/s**
- variable order of filling the cells — **240 000 DLS/s**
- special program implementation with $N^2$ nested loops — **340 000 DLS/s**
- use the principal of minimum abilities — **790 000 DLS/s**
- clipping for selected depth only — **1 100 000 DLS/s**
- use the formulas for magic squares — **1 800 000 DLS/s**
- use the bits arithmetic magic — **6 600 000 DLS/s** (2016)

Pace increased by **8 orders** using algorithmic optimization without parallelization!

Pace can be additionally increased using SCFs...

## Results: number of DLS of order N<10

A274171 (Number of diagonal Latin squares of order n with first row 1..n)

**1, 0, 0, 2, 8, 128, 171200, 7447587840, 5056994653507584**

A274806 (Number of diagonal Latin squares of order n)

**1, 0, 0, 48, 960, 92160, 862848000, 300286741708800, 1835082219864832081920**

$$L_{10} \simeq \left(7{,}6 \div 10{,}9\right) \cdot 10^{22}$$

~250 000 years at Gerasim@Home distributed computing project

~1 year at 1 PFLOP/s supercomputer (who can help us? :) )

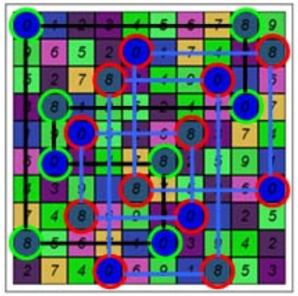
- Gerasim@Home (~500 PCs, ~3 months, 2–5 TFLOP/s), http://gerasim.boinc.ru
- Matrosov academician computing cluster (~500 24/7 CPU cores, ~3 months)

## Classification of DLS of order 10 by number of orthogonal squares

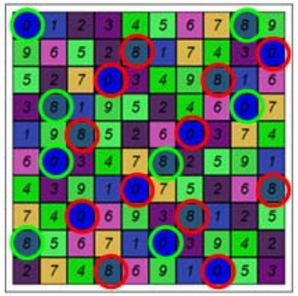
From all ODLS CFs we have (on 02.04.2017):

- **1:0** — absolute majority (~1 by 30 000 000 DLS, bachelor)
- **1:1** — >400 000 (most of known ODLS)
- **1:2** — >4 100 (symmetric and not!)
- **1:3** — 4 (whitefox, progger, 2017)
- **1:4** — 209
- **1:6** — 6
- **1:8** — 4
- some else?...

XO = 12



| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 2 | 0 | 4 | 3 | 6 | 9 | 8 | 5 | 7 |
| 2 | 7 | 8 | 6 | 9 | 3 | 5 | 1 | 0 | 4 |
| 8 | 0 | 5 | 7 | 1 | 9 | 2 | 3 | 4 | 6 |
| 3 | 8 | 6 | 9 | 5 | 7 | 1 | 4 | 2 | 0 |
| 5 | 9 | 7 | 1 | 0 | 4 | 8 | 6 | 3 | 2 |
| 7 | 6 | 4 | 8 | 2 | 1 | 3 | 0 | 9 | 5 |
| 4 | 5 | 3 | 0 | 6 | 2 | 7 | 9 | 1 | 8 |
| 9 | 4 | 1 | 5 | 7 | 8 | 0 | 2 | 6 | 3 |
| 6 | 3 | 9 | 2 | 8 | 0 | 4 | 5 | 7 | 1 |



| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 2 | 3 | 7 | 0 | 9 | 8 | 5 | 4 | 6 |
| 4 | 0 | 9 | 6 | 3 | 7 | 1 | 8 | 2 | 5 |
| 9 | 6 | 8 | 4 | 5 | 1 | 3 | 0 | 7 | 2 |
| 5 | 9 | 6 | 8 | 7 | 0 | 2 | 4 | 3 | 1 |
| 3 | 4 | 5 | 9 | 2 | 8 | 0 | 6 | 1 | 7 |
| 8 | 7 | 0 | 1 | 6 | 3 | 5 | 2 | 9 | 4 |
| 2 | 3 | 7 | 5 | 9 | 6 | 4 | 1 | 0 | 8 |
| 7 | 5 | 1 | 2 | 8 | 4 | 9 | 3 | 6 | 0 |
| 6 | 8 | 4 | 0 | 1 | 2 | 7 | 9 | 5 | 3 |

«Treshka» from whitefox

Different types of classification are under development (fish, rhombus, ...), classification process needs to be automated!

# GPU implementation of Euler-Parker approach (I. Shutov)

10x times faster than single threaded CPU implementation

Based on:

- parallel processing of different squares on different SMXs
- parallel building of sets of transversals based on 10 cells (with WARP)
- efficient use of the CUDA shared memory (for square being processed) and register memory (for additional data structures)

Advantages and disadvantages:

- faster than single threaded CPU (~2500 CUDA cores per GPU working in parallel!)
- slower that peak abilities of GPU (for example, molecular dynamics or N-body problem — 600x times faster than CPU)

Problems:

- Recursive algorithm (but iterative implementation)
- irregular if's patterns (difficult to effective execution of WARPs)
- irregular memory accesses

# Dancing Links X algorithm (DLX, D. Knuth, 2000)

At now 4x times faster than bits arithmetic approach

Based on fast decision of the **exact cover problem** solving

Generating of DLS —

$N^3 \times 3N^2+2N$

Generating of normalized DLS —

$N^3–N^2 \times 3N^2+2N$

Transversals set building —

$N^2 \times 3N+2$

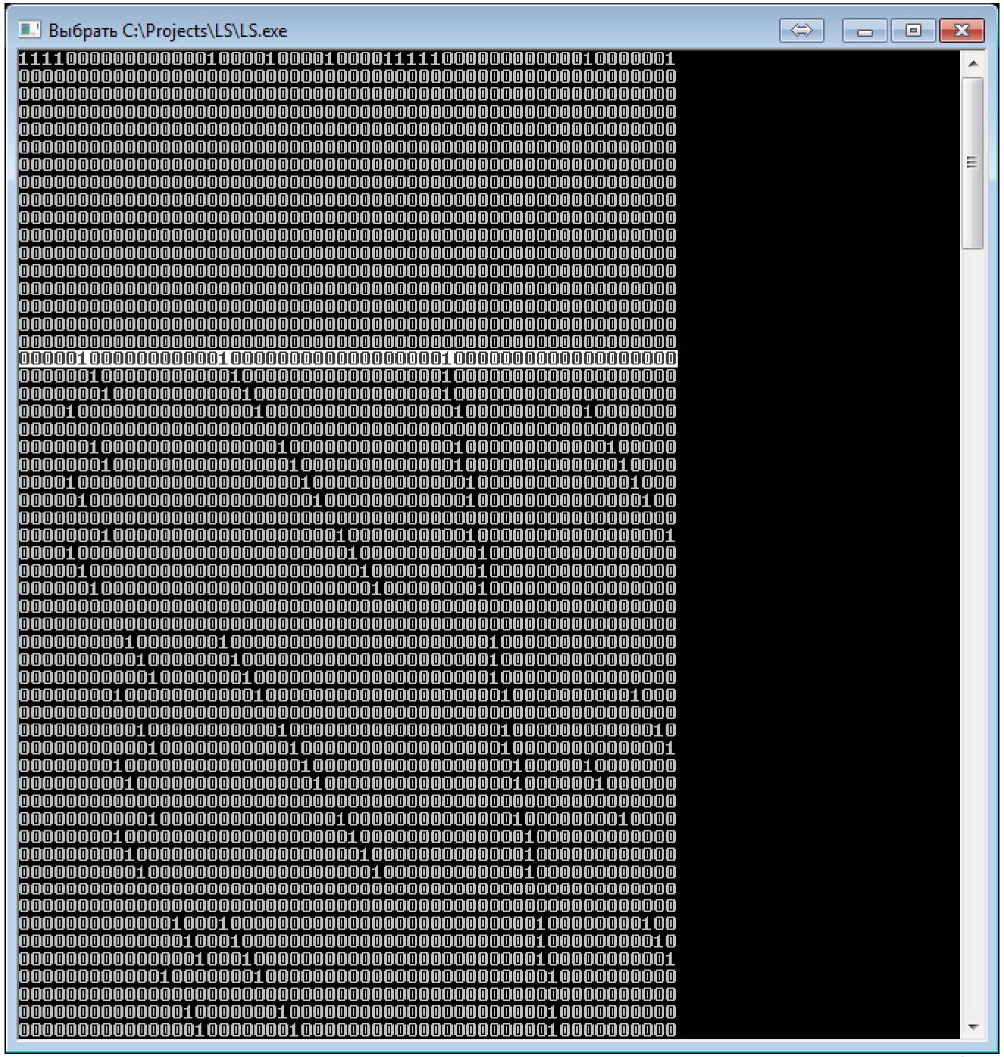
Getting of ODLS (directly) —

$N^3 \times 4N^2+2N$

Getting of ODLS using transversals set (efficient) —

$T \times N^2$

Can be implemented on GPU? Shared memory volume restriction? Irregular memory access patterns?



# Strong normalized DLS

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | 2 | | | | | | | 0 | |
| | | 1 | | | | | 2 | | |
| | | | 4 | | | 1 | | | |
| | | | | 3 | 4 | | | | |
| | | | | 5 | 6 | | | | |
| | | | 6 | | | 5 | | | |
| | | 3 | | | | | 8 | | |
| | 7 | | | | | | | 9 | |
| 8 | | | | | | | | | 7 |

**2 723 433 984** different fillings
(huge amount)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | 7 |
| | 1 | | | | | | | 0 | |
| | | 2 | | | | | 1 | | |
| | | | 3 | | | 2 | | | |
| | | | | 4 | 3 | | | | |
| | | | | 6 | 5 | | | | |
| | | | 5 | | | 6 | | | |
| | | 4 | | | | | 7 | | |
| | 9 | | | | | | | 8 | |
| 8 | | | | | | | | | 9 |

**440 192** different fillings only
(~ 6000x times less)

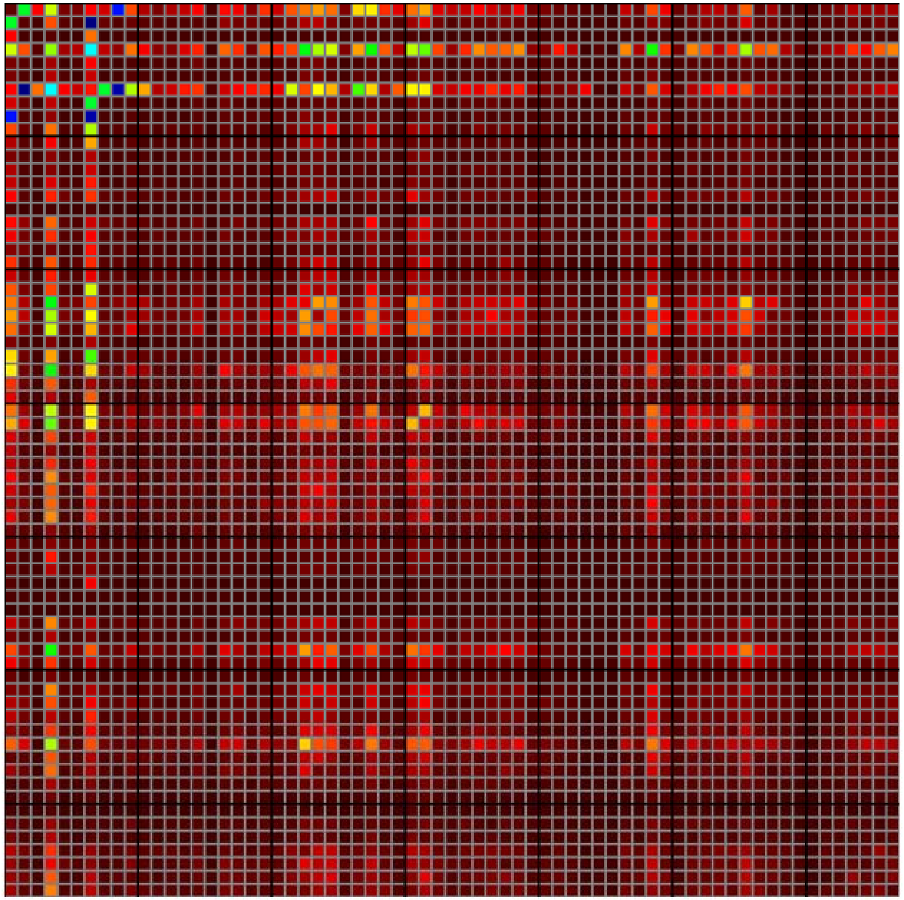
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | 1 |
| | 1 | | | | | | | 0 | |
| | | 2 | | | | | 3 | | |
| | | | 3 | | | 2 | | | |
| | | | | 4 | 6 | | | | |
| | | | | 8 | 5 | | | | |
| | | | 7 | | | 6 | | | |
| | | 4 | | | | | 7 | | |
| | 9 | | | | | | | 8 | |
| 5 | | | | | | | | | 9 |

**67** different fillings only!!!
(~ 6500x times less)

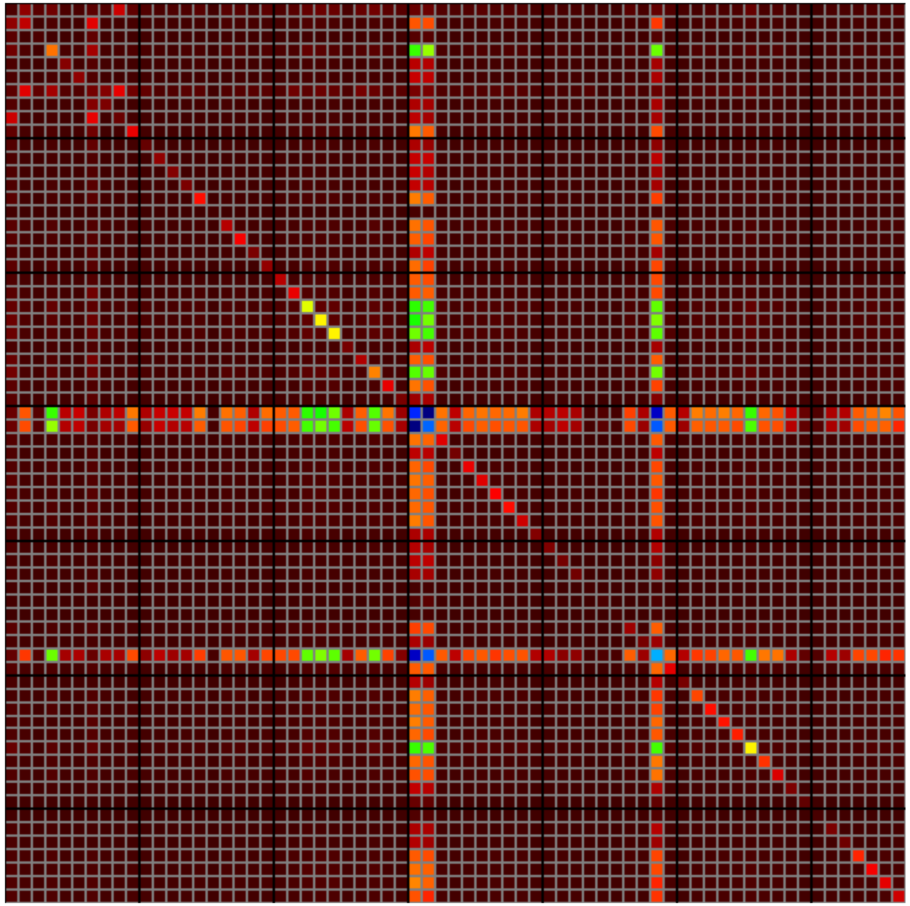
- ODLS@Home, Gerasim@Home – analyzing different lines (some interesting features during canonization DLS by LS)

- 67 lines of SCFs with different properties (multiplicity, CFs density, ODLS CFs density, LS-by-DLS canonization features)

## Cross correlation of SCFs



**Gerasim@Home, ~ 60 000 CFs, random + symmetry + Browns**



**Total, ~ 230 000 CFs**

- ~ 40 hours for calculating, requires database with ODLS properties
- different multiplicity of lines
- SODLS

## Separating of SCFs to isomorphism classes

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | | | | | | | 1 |
| | 1 | | | | | 0 | |
| | | 2 | | | 3 | | |
| | | | 3 | 2 | | | |
| | | | 5 | 4 | | | |
| | | 4 | | | 5 | | |
| | 7 | | | | | 6 | |
| 6 | | | | | 0 | | 7 |

**213 920** different DLSs

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | | | | | | | 1 |
| | 1 | | | | | 0 | |
| | | 2 | | | 3 | | |
| | | | 3 | 2 | | | |
| | | | 5 | 4 | | | |
| | | 4 | | | 5 | | |
| | 7 | | | 3 | | 6 | |
| 6 | | | | | | | 7 |

**213 920** different DLSs
also, why?
Same isomorphism class!

- require to change order of filling of lexicographic strings (simple, but CFs are differ);
- require to work with partially filled CFs (new algorithm, implementation, optimization);
- can be used for fast enumerating (10x – 100x times faster).

## Related work

Collecting CFs and new combinatorial structures search:

- triple of MODLS (is it exist?)
- different structures?

GPU implementation of transversal and cover algorithms:

- Euler-Parker approach – need to deploy to Gerasim@Home project
- DLX – need to develop (it is faster on CPU, and what about GPU?)

Enumeration problems (OEIS):

- expanding current sequences
- enumerating DLS and ODLS of special kind (string-inverse, symmetric, ...) and its CFs
- special procedures for special types of squares (symmetric DLX, ...)

Pseudo triples:

- 3 kinds of pseudo triples, only 1 was investigated in details
- special approaches (optimization problem, ACO?, ...)

# Thank you for your attention!

Thanks to all the volunteers who took part in the Gerasim@home project!

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