Southwest State University
Department of computer sciences

# ENUMERATING CYCLIC AND PANDIAGONAL LATIN SQUARES, CORRESPONDING MAIN CLASSES AND THEIR PROPERTIES

**Vatutin E.I.**

Kursk, 2021

## What are Latin and diagonal Latin squares?

$$A = \left\| a_{ij} \right\|$$

$$i,\ j = \overline{1,\ N}$$

$$N = |S|$$

$$S = \{0, 1, 2, \ldots, N-1\}$$

$$\forall i,\ j,\ k = \overline{1,\ N},\ j \neq k : \left(a_{ij} \neq a_{ik}\right) \wedge \left(a_{ji} \neq a_{ki}\right)$$

$$\forall i,\ j = \overline{1,\ N},\ i \neq j : \left(a_{ii} \neq a_{jj}\right) \wedge \left(a_{N-i+1,\ N-i+1} \neq a_{N-j+1,\ N-j+1}\right)$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 9 & 4 & 3 & 6 & 7 & 5 & 0 & 8 \\ 2 & 9 & 3 & 1 & 7 & 0 & 5 & 8 & 4 & 6 \\ 3 & 4 & 1 & 2 & 8 & 7 & 9 & 6 & 5 & 0 \\ 4 & 3 & 5 & 9 & 2 & 1 & 8 & 0 & 6 & 7 \\ 5 & 6 & 4 & 8 & 1 & 2 & 0 & 9 & 7 & 3 \\ 6 & 5 & 8 & 7 & 0 & 3 & 2 & 1 & 9 & 4 \\ 7 & 8 & 6 & 0 & 9 & 4 & 1 & 2 & 3 & 5 \\ 8 & 7 & 0 & 5 & 6 & 9 & 3 & 4 & 1 & 2 \\ 9 & 0 & 7 & 6 & 5 & 8 & 4 & 3 & 2 & 1 \end{pmatrix}$$

Normalized LS of order 10

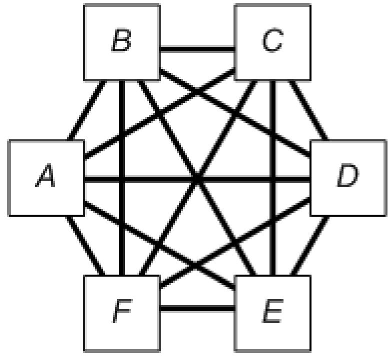
$$N! \times (N-1)!$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 4 & 9 & 0 & 6 & 5 & 1 & 3 & 8 \\ 8 & 3 & 6 & 7 & 5 & 9 & 0 & 2 & 4 & 1 \\ 2 & 6 & 8 & 5 & 1 & 7 & 4 & 0 & 9 & 3 \\ 5 & 8 & 9 & 1 & 7 & 0 & 3 & 4 & 6 & 2 \\ 9 & 4 & 1 & 2 & 8 & 3 & 7 & 6 & 0 & 5 \\ 4 & 7 & 5 & 6 & 9 & 1 & 8 & 3 & 2 & 0 \\ 3 & 0 & 7 & 8 & 2 & 4 & 1 & 9 & 5 & 6 \\ 6 & 5 & 0 & 4 & 3 & 2 & 9 & 8 & 1 & 7 \\ 1 & 9 & 3 & 0 & 6 & 8 & 2 & 5 & 7 & 4 \end{pmatrix}$$

Normalized DLS of order 10

$$(N-1)!$$

# Why is it interesting?




Applied problems:

- experiment planning
- cryptography
- error correcting codes
- scheduling
- algebra, combinatorics, statistics, …

Mathematical problems:

- **existence of a triple of MOLS/MODLS of order 10 (or larger clique)**
- increasing world record of orthogonality characteristic for pseudo triple of MOLS (291/300) or MODLS (274/300)
- generating functions
- asymptotic behavior of combinatorial characteristics based on DLSs (OEIS)
- number theory (relations between different fields of knowledge)
- magic squares
- Sudoku (LS of order 9 with additional constraints)

# Special types of LS/DLS

row[i] = cyclic_shift(row[i-1], d)

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 6 | 0 | 1 | 2 | 3 | 4 | 5 |

Cyclic LS (d=1)

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 6 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 5 | 6 | 0 | 1 | 2 | 3 | 4 |

Cyclic DLS (d=2)

- there are known different special types of LS/DLS (plane symmetric, central symmetric, …);
- all cyclic squares of small orders (N<13) are pandiagonal.

# Integer sequences (OEIS): N queens problem example

*The OEIS Foundation is supported by donations from users of the OEIS and by a grant from the Simons Foundation.*

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

n queens Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **n queens**

Displaying 1-10 of 708 results found. page 1 2 3 4 5 6 7 8 9 10 ... 71

Sort: relevance | references | number | modified | created Format: long | short | data

A000170 Number of ways of placing **n** nonattacking **queens** on an **n** X **n** board. +40 75

(Formerly M1958 N0775)

1, 1, 0, 0, 2, 10, 4, 40, 92, 352, 724, 2680, 14200, 73712, 365596, 2279184, 14772512, 95815104, 666090624, 4968057848, 39029188884, 314666222712, 2691008701644, 24233937684440, 227514171973736, 2207893435808352, 22317699616364044, 234907967154122528 (list; graph; refs; listen; history; edit; text; internal format)

OFFSET 0,5

COMMENTS For **n** > 3, a(**n**) is the number of maximum independent vertex sets in the **n** X **n** **queen** graph. - Eric W. Weisstein, Jun 20 2017

Number of nodes on level **n** of the backtrack tree for the **n** **queens** problem (a(**n**) = A319284(**n**, **n**)). - Peter Luschny, Sep 18 2018

REFERENCES M. Gardner, The Unexpected Hanging, pp. 190-2, Simon & Shuster NY 1969

Jieh Hsiang, Yuh-Pyng Shieh and Yao-Chiang Chen, The cyclic complete mappings counting problems, in Problems and Problem Sets for ATP, volume 02-10 of DIKU technical reports, G. Sutcliffe, J. Pelletier and C. Suttner, eds., 2002.

D. E. Knuth, The Art of Computer Programming, Volume 4, Pre-fascicle 5B, Introduction to Backtracking, 7.2.2. Backtrack programming. 2018.

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W. W. Rouse Ball and H. S. M. Coxeter, Mathematical Recreations and Essays, 13th ed., New York, Dover, 1987, pp. 166-172 (The Eight **Queens** Problem).

M. A. Sainte-Laguë, Les Réseaux (ou Graphes), Mémorial des Sciences Mathématiques, Fasc. 18, Gauthier-Villars, Paris, 1926, p. 47.

**N.** J. A. Sloane, A Handbook of Integer Sequences, Academic Press, 1973 (includes this sequence).

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R. J. Walker, An enumerative technique for a class of combinatorial problems, pp. 91-94 of Proc. Sympos. Applied Math., vol. 10, Amer. Math. Soc., 1960.

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LINKS Table of n, a(n) for n=0..27.

Jordan Bell, Brett Stevens, A survey of known results and research areas for n-**queens**, Discrete Mathematics, Volume 309, Issue 1, Jan 06 2009, Pages 1-31.

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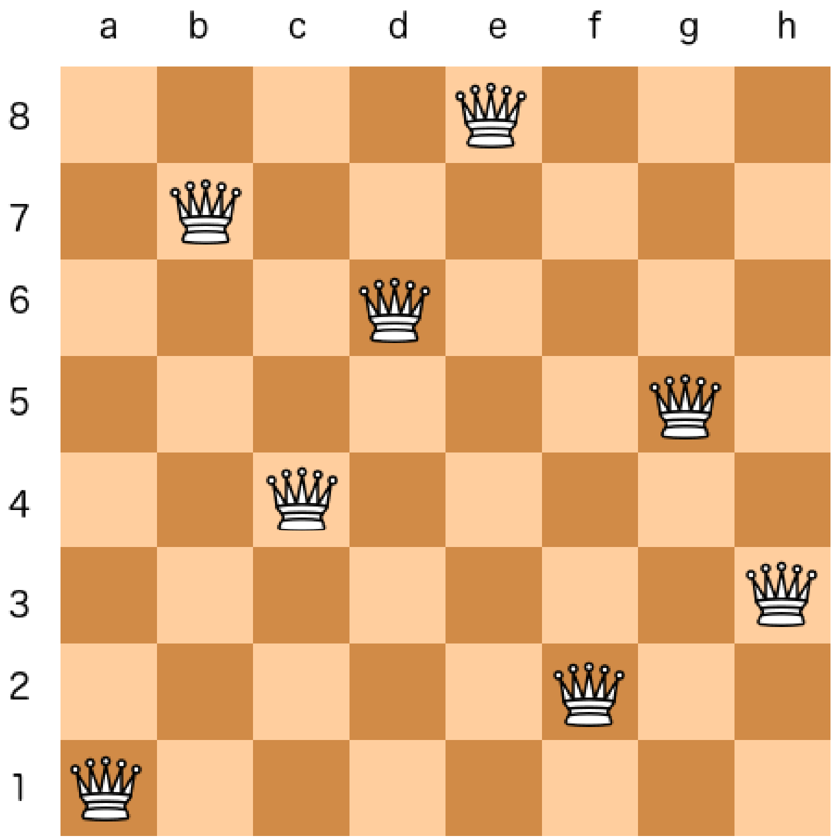
J. R. Bitner and E. M. Reingold, Backtrack programming techniques, Commun. ACM, 18 (1975), 651-656. [Annotated scanned copy]

P. Capstick and K. McCann, The problem of the **n** **queens**, apparently unpublished, no date (circa 1990?) [Scanned copy]

V. Chvatal, All solutions to the problem of eight **queens**




https://oeis.org/A000170

## Integer sequences connected with transversals in DLS

**A287645** — Minimum number of transversals in a diagonal Latin square of order N (N<10)

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**A287647** — Minimum number of diagonal transversals in a diagonal Latin square of order N (N<9)

**A287648** — Maximum number of diagonal transversals in a diagonal Latin square of order N (N<9)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 3 | | | | | | | | | |
| | | | 7 | | | | | | |
| | | | | | 2 | | | | |
| | | | | 0 | | | | | |
| | | | | | | | 6 | | |
| | | | | | | | | 5 | |
| | | | | | | | | | 4 |
| | | | | | | 8 | | | |
| | 9 | | | | | | | | |
| | | 1 | | | | | | | |

Transversal 1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 3 | | | | | | | | | |
| | | | 7 | | | | | | |
| | | | | | | | | | 6 |
| | 8 | | | | | | | | |
| | | | | | 4 | | | | |
| | | | | | | | | 5 | |
| | | | | | | | 2 | | |
| | | | | 9 | | | | | |
| | | | | | | 0 | | | |
| | | 1 | | | | | | | |

Transversal 2

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | | | | | | | | |
| | | | | | | 3 | | | |
| | | | | | | | | | 6 |
| | | | | | 9 | | | | |
| | | 7 | | | | | | | |
| | | | | 1 | | | | | |
| 0 | | | | | | | | | |
| | | | | | | | 4 | | |
| | | | 5 | | | | | | |
| | | | | | | | | 8 | |

Transversal 3

## Integer sequences connected with transversals in DLS: exactly known values

**A287645** — **1, 0, 0, 8, 3, 32, 7, 8**, **68** (N<10)
**A287644** — **1, 0, 0, 8, 15, 32, 133, 384**, **2241** (N<10)
**A287647** — **1, 0, 0, 4, 1, 2, 0, 0**, **0** (N<10)
**A287648** — **1, 0, 0, 4, 5, 6, 27, 120**, **333** (N<10)

a(1)-a(8) — Brute Force for all DLS
a(9) — CFs generator for main classes of DLS based on X-fillings of diagonals and ESODLS schemas (currently ongoing, ends)

What about a(10), a(11), …?
Precise values are unknown, only upper and lower bounds are known.

$$0 \le X_{\min}^{LS}(N) \le X_{\min}^{DLS}(N) \le X_{\max}^{DLS}(N) \le X_{\max}^{LS}(N)$$

## Integer sequences connected with transversals in DLS: unknown values for high orders

**A287645** — 1, 0, 0, 8, 3, 32, 7, 8, 68, **≤408, ≤2477, ≤2240, ≤78253, ≤422312, ≤2415635, ≤14689972** (0 ≤ a(n) ≤ A287644(n) ≤ A090741(n))

**A287644** — 1, 0, 0, 8, 15, 32, 133, 384, 2241, **≥5504, ≥37851, ≥198144, ≥1030367, ≥428296, ≥2991104?, ≥2429398, ≥3340285?, ≥14720910, ≥244744192?, ≥1606008513, ≥2167746304?, ≥87656896891, ≥697292390400?, ≥51162162017?, ≥?, ≥452794797220965, ≥?, ≥41609568918940625** (a(n) <= A090741(n))

**A287647** — 1, 0, 0, 4, 1, 2, 0, 0, 0, **≤15, ≤279, ≤74, ≤8795, ≤52484?, ≤?, ≤3994676, ≤204330233, ≤?, ≤11232045257** (a(n) <= A287648(n) <= A007016(n))

**A287648** — 1, 0, 0, 4, 5, 6, 27, 120, 333, **≥866, ≥4828, ≥30192, ≥131106, ≥380718, ≥389318, ≥32172800, ≥204995269, ≥280308432, ≥11254190082, ≥90010806304, ≥51162162017, ≥322747329246** (a(n) <= A007016(n))

Upper and lower bounds can be expanded and strengthened!

# Transversals in random Latin squares

A287645 (minimum number of transversals)
a(11) ≤ 3266->3255->3185->3175
a(12) ≤ 15462
a(13) ≤ 78253
a(14) ≤ 422312
a(15) ≤ 2415635
a(16) ≤ 14689972
...

A287644 (maximum number of transversals)
a(11) ≥ 3867
a(12) ≥ 16600
a(13) ≥ 80999 (a(13) ≥ 1030367 for cyclic squares)
a(14) ≥ 428296
a(15) ≥ 2429398
a(16) ≥ 14720910
...

- 16 days on Core i7 4770 CPU, special program implementation of DLX (enumeration only without covers collecting)
- https://vk.com/wall162891802_1449

# Diagonal transversals in random Latin squares

A287647 (minimum number of diagonal transversals)
a(11) ≤ 324
a(12) ≤ 1816
...

A287648 (maximum number of diagonal transversals)
a(11) ≥ 550
a(12) ≥ 2200
...

- 16 days on Core i7 4770 CPU, special program implementation of DLX (enumeration only without covers collecting)
- https://vk.com/wall162891802_1449

# Cyclic Latin squares: one of special types of LS

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

d=0

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 6 | 0 | 1 | 2 | 3 | 4 | 5 |

d=1

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 6 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 5 | 6 | 0 | 1 | 2 | 3 | 4 |

d=2

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 6 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 4 | 5 | 6 | 0 | 1 | 2 | 3 |

d=3

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 6 | 0 | 1 | 2 | 3 | 4 | 5 |
| 3 | 4 | 5 | 6 | 0 | 1 | 2 |

d=4

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 6 | 0 | 1 | 2 | 3 | 4 | 5 |
| 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 2 | 3 | 4 | 5 | 6 | 0 | 1 |

d=5

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 6 | 0 | 1 | 2 | 3 | 4 | 5 |
| 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 1 | 2 | 3 | 4 | 5 | 6 | 0 |

d=6

- 4 cyclic DLS, 6 cyclic LS for N=7

# Number of transversals in cyclic Latin squares

1, 0, 3, 0, 15, 0, 133, 0, 2025, 0, 37851, 0, 1030367, 0, 36362925

or without zeroes:

1, 3, 15, 133, 2025, 37851, 1030367, 36362925

A006717 — Number of ways of arranging 2n+1 nonattacking semi-queens on a (2n+1) X (2n+1) toroidal board

Also the number of transversals of a cyclic Latin square of order 2n+1 and the number of orthomorphisms of the cyclic group of order 2n+1 (Ian Wanless, 2001).

- Number of transversals are same for all cyclic LS of given order N

## Number of transversals in cyclic Latin squares: consequence

A090741 (maximum number of transversals in LS):

$a(11) \geq 37851$
$a(13) \geq 1030367$
$a(15) \geq 36362925$
$a(17) \geq 1606008513$
$a(19) \geq 87656896891$
$a(21) \geq 5778121715415$
$a(23) \geq 452794797220965$
$a(25) \geq 41609568918940625$
...

Lower bounds can be added to sequence A090741 in OEIS...

- Number of transversals are same for all cyclic LS of given order N

# Number of transversals in cyclic diagonal Latin squares: consequence

a(11) ≥ 37851
a(13) ≥ 1030367
cyclic DLS are not exists for N=15
a(17) ≥ 1606008513
a(19) ≥ 87656896891
cyclic DLS are not exists for N=21
a(23) ≥ 452794797220965
a(25) ≥ 41609568918940625

Lower bounds can be added to sequence A287644 in OEIS...

- exists not for all orders N

## Number of diagonal transversals in cyclic diagonal Latin squares

Minimal number of diagonal transversals in cyclic DLS of order N=2n+1: **<u>1, 0, 5, 27, 0, 4523, 128818, 0, 204330233, 11232045257</u>** (not presented in OEIS, added as A342998)

Maximal number of diagonal transversals in cyclic DLS of order N=2n+1 : **<u>1, 0, 5, 27, 0, 4665, 131106, 0, 204995269, 11254190082</u>** (not presented in OEIS, added as A342997)

Corresponding upper and lower bounds can be added to A287647 (weak) and A287648...

## Enumerating the cyclic (diagonal) Latin squares

DLS:

**1, 0, 0, 0, 2, 0, 4, 0, 0, 0, 8, 0, 10, 0, 0, 0, 14, 0, 16, 0, 0, 0, 20, 0, 10, 0, 0, 0, 26, 0, 28, …**

Squares of this special type exists not for all orders N.

The number of positive integers k which are ≤ n and where k, k-1 and k+1 are each coprime to n (well known numerical series that directly not connected with Latin squares!).

LS:

**1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, 8, 8, 16, 6, 18, 8, …**

Euler totient function phi(N)!!!
Not connected directly with Latin squares! Can be calculated through Latin squares…

## Euler totient function calculating

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_m^{\alpha_m}$$

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_m}\right)$$

$$6 = 2^1 \cdot 3^1,\ \varphi(6) = 6\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 2$$

- calculating by definition (enumerating coprimes (k,n), k<n, Euclid algorithm)
- calculating using factoring and formula
- new method: calculating by enumerating cyclic Latin squares of order n

## Euler totient function calculating by enumerating cyclic Latin squares

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | 2 | 3 | 4 | 5 |

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 0 | 1 | 2 | 3 | 4 |

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 4 | 5 | 0 | 1 | 2 | 3 |

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 |
| 3 | 4 | 5 | 0 | 1 | 2 |

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 0 | 1 | 2 | 3 | 4 | 5 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 2 | 3 | 4 | 5 | 0 | 1 |

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 5 | 0 | 1 | 2 | 3 | 4 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 1 | 2 | 3 | 4 | 5 | 0 |

$$\varphi(6) = 2$$

- polynomial algorithm ($t \sim O(N^2)$, $m \sim O(N)$) without multiplications and divisions

# Euler totient function OEIS description slightly expanded

```
phi(p*n) = phi(n)*(floor(((n + p - 1) mod p)/(p - 1)) + p - 1), for primes p. -
  Gary Detlefs, Apr 21 2012
For odd n, a(n) = 2*A135303((n-1)/2)*A003558((n-1)/2) or phi(n) = 2*c*k; the Coach
  theorem of Pedersen et al. Cf. A135303. - Gary W. Adamson, Aug 15 2012
G.f.: Sum_{n>=1} mu(n)*x^n/(1 - x^n)^2, where mu(n) = A008683(n). - Mamuka
  Jibladze, Apr 05 2015
a(n) = n - cototient(n) = n - A051953(n). - Omar E. Pol, May 14 2016
a(n) = lim_{s->1} n*zeta(s)*(Sum_{d divides n} A008683(d)/(e^(1/d))^(s-1)), for n >
  1. - Mats Granvik, Jan 26 2017
Conjecture: a(n) = Sum_{a=1..n} Sum_{b=1..n} Sum_{c=1..n} 1 for n > 1. The sum is
  over a,b,c such that n*c - a*b = 1. - Benedict W. J. Irwin, Apr 03 2017
a(n) = Sum_{j=1..n} gcd(j, n) cos(2*Pi*j/n) = Sum_{j=1..n} gcd(j, n)
  exp(2*Pi*i*j/n) where i is the imaginary unit. Notice that the Ramanujan's sum
  c_n(k) := Sum_{j=1..n, gcd(j, n) = 1} exp(2*Pi*i*j*k/n) gives a(n) = Sum_{k|n}
  k*c_(n/k)(1) = Sum_{k|n} k*mu(n/k). - Michael Somos, May 13 2018
G.f.: x*d/dx(x*d/dx(log(Product_{k>=1} (1 - x^k)^(-mu(k)/k^2)))), where mu(n) =
  A008683(n). - Mamuka Jibladze, Sep 20 2018
a(n) = Sum_{d|n} A007431(d). - Steven Foster Clark, May 29 2019
G.f. A(x) satisfies: A(x) = x/(1 - x)^2 - Sum_{k>=2} A(x^k). - Ilya Gutkovskiy, Sep
  06 2019
a(n) >= sqrt(n/2) (Nicolas). - Hugo Pfoertner, Jun 01 2020
a(n) > n/(exp(gamma)*log(log(n)) + 5/(2*log(log(n)))), except for n=223092870
  (Rosser, Schoenfeld). - Hugo Pfoertner, Jun 02 2020
```

EXAMPLE

```
G.f. = x + x^2 + 2*x^3 + 2*x^4 + 4*x^5 + 2*x^6 + 6*x^7 + 4*x^8 + 6*x^9 + 4*x^10 +
  ...
a(8) = 4 with {1, 3, 5, 7} units modulo 8. a(10) = 4 with {1, 3, 7, 9} units modulo
  10. - Michael Somos, Aug 27 2013
From Eduard I. Vatutin, Nov 01 2020: (Start)
The a(5)=4 cyclic Latin squares with the first row in ascending order are:
  0 1 2 3 4   0 1 2 3 4   0 1 2 3 4   0 1 2 3 4
  1 2 3 4 0   2 3 4 0 1   3 4 0 1 2   4 0 1 2 3
  2 3 4 0 1   4 0 1 2 3   1 2 3 4 0   3 4 0 1 2
  3 4 0 1 2   1 2 3 4 0   4 0 1 2 3   2 3 4 0 1
  4 0 1 2 3   3 4 0 1 2   2 3 4 0 1   1 2 3 4 0
(End)
```

MAPLE

```
with(numtheory): A000010 := phi; [ seq(phi(n), n=1..100) ]; # version 1
with(numtheory): phi := proc(n) local i, t1, t2; t1 := ifactors(n)[2]; t2 :=
  n*mul((1-1/t1[i][1]), i=1..nops(t1)); end; # version 2
```

MATHEMATICA

```
Array[EulerPhi, 70]
```

PROG

```
(Axiom) [eulerPhi(n) for n in 1..100]
(MAGMA) [ EulerPhi(n) : n in [1..100] ]; // Sergei Haller (sergei(AT)sergei-
  haller.de), Dec 21 2006
(PARI) {a(n) = if( n==0, 0, eulerphi(n))}; /* Michael Somos, Feb 05 2011 */
(Sage)
# euler_phi is a standard function in Sage.
def A000010(n): return euler_phi(n)
def A000010_list(n): return [ euler_phi(i) for i in range(1, n+1)]
# Jaap Spies, Jan 07 2007
(PARI) { for (n=1, 100000, write("b000010.txt", n, " ", eulerphi(n))); } \\ Harry
  J. Smith, Apr 26 2009
(Sage) [euler_phi(n) for n in range(1, 70)]  # Zerinvary Lajos, Jun 06 2009
(Maxima) makelist(totient(n), n, 0, 1000); /* Emanuele Munarini, Mar 26 2011 */
(Haskell) a n = length (filter (==1) (map (gcd n) [1..n])) -- Allan C. Wechsler,
  Dec 29 2014
(Python)
from sympy.ntheory import totient
print([totient(i) for i in range(1, 70)])  # Indranil Ghosh, Mar 17 2017
```

CROSSREFS

```
Cf. A008683, A003434 (steps to reach 1), A007755, A049108, A002202 (values).
Cf. A005277 (nontotient numbers). For inverse see A002181, A006511, A058277.
Jordan function J_k(n) is a generalization - see A059379 and A059380 (triangle of
  values of J_k(n)), this sequence (J_1), A007434 (J_2), A059376 (J_3), A059377
```

# Euler totient function calculating by enumerating cyclic Latin squares: practice implementation

```
C:\Projects\Функция Эйлера\EulerTotientFunction.exe
1 - factorization
2 - Euclid
3 - cyclic Latin squares
4 - cyclic Latin squares with 0 find
5 - cyclic Latin squares with 0 find, half of d values

   N   v1     v2     v3     v4     v5   t1    t2     t3     t4     t5
--------------------------------------------------------------------------
   1:   1 -    1 -    1 -    1 -    1  221    94    227    175    287
   2:   1 -    1 -    1 -    1 -    1   54   166     79    154     66
   3:   2 -    2 -    2 -    2 -    2  166   605    284    163     75
   4:   2 -    2 -    2 -    2 -    2  217   314    568    257     76
   5:   4 -    4 -    4 -    4 -    4  211   613    858    475    311
   6:   2 -    2 -    2 -    2 -    2  166   456   1007    583    266
   7:   6 -    6 -    6 -    6 -    6  221   846   1236    749    299
   8:   4 -    4 -    4 -    4 -    4  239   843   1438    783    308
   9:   6 -    6 -    6 -    6 -    6  229   873   1991    810    553
  10:   4 -    4 -    4 -    4 -    4  200   822   2019    916    447
  11:  10 -   10 -   10 -   10 -   10  263  1257   3255   1472    774
  12:   4 -    4 -    4 -    4 -    4  211   955   2575    949    501
  13:  12 -   12 -   12 -   12 -   12  145  1441   4198   1638   1161
  14:   6 -    6 -    6 -    6 -    6  145  1324   3745   1641    907
  15:   8 -    8 -    8 -    8 -    8  202  1360   3974   1508   1085
  16:   8 -    8 -    8 -    8 -    8  284  1457   4896   2004   1173
  17:  16 -   16 -   16 -   16 -   16  212  1514   6624   2427   1753
  18:   6 -    6 -    6 -    6 -    6  378  1992   5500   1988   1257
  19:  18 -   18 -   18 -   18 -   18  221  2191   8758   3168   2224
  20:   8 -    8 -    8 -    8 -    8  305  1913   6519   2370   1481
  21:  12 -   12 -   12 -   12 -   12  339  2433   8151   3044   1777
  22:  10 -   10 -   10 -   10 -   10  260  2125   8904   3197   2076
  23:  22 -   22 -   22 -   22 -   22  281  2838  12497   4231   2705
  24:   8 -    8 -    8 -    8 -    8  227  2294   8471   2935   1892
  25:  20 -   20 -   20 -   20 -   20  305  2898  12745   4352   2744
  26:  12 -   12 -   12 -   12 -   12  184  2877   9632   4558   2557
  27:  18 -   18 -   18 -   18 -   18  336  2844  10330   5129   2641
  28:  12 -   12 -   12 -   12 -   12  284  3095  10043   4512   2412
  29:  28 -   28 -   28 -   28 -   28  242  3660  14609   7220   3935
  30:   8 -    8 -    8 -    8 -    8  580  3602  12497   4820   2880
  31:  30 -   30 -   30 -   30 -   30  260  3911  21007   7259   4642
  32:  16 -   16 -   16 -   16 -   16  353  3551  17396   6108   4104
  33:  20 -   20 -   20 -   20 -   20  227  3802  19527   6616   3959
  34:  16 -   16 -   16 -   16 -   16  199  3975  20179   7265   4364
  35:  24 -   24 -   24 -   24 -   24  263  3835  21207   7193   4382
  36:  12 -   12 -   12 -   12 -   12  408  3926  17393   5794   3666
  37:  36 -   36 -   36 -   36 -   36  411  4703  27321  10110   6274
  38:  18 -   18 -   18 -   18 -   18  314  4540  19013   8792   5237
  39:  24 -   24 -   24 -   24 -   24  320  4721  20893   8716   5098
  40:  16 -   16 -   16 -   16 -   16  241  4497  22573   7700   4929
  41:  40 -   40 -   40 -   40 -   40  124  5349  35158  12189   7942
  42:  12 -   12 -   12 -   12 -   12  348  4706  23081   7713   4724
  43:  42 -   42 -   42 -   42 -   42  173  5213  38388  13023   8015
  44:  20 -   20 -   20 -   20 -   20  375  5141  28702  70279   5939
  45:  24 -   24 -   24 -   24 -   24  387  5364  80872  10152   6573
  46:  22 -   22 -   22 -   22 -   22  326  5751  80826  11729   7308
  47:  46 -   46 -   46 -   46 -   46  263  6253  91978  15556   9559
  48:  16 -   16 -   16 -   16 -   16  405  5558  66552  10185   6253
  49:  42 -   42 -   42 -   42 -   42  438  6422  90340  15226   9278
  50:  20 -   20 -   20 -   20 -   20  565  6495  80645  12095   7477
```

- slower than factorization based calculating
- slower than long arithmetic implementation (extimation)

## Euler totient function calculating by enumerating cyclic Latin squares and main classes (is it possible?)

Pandiagonal LS of order N (added to OEIS as A339999):

**1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 2, 0**

Cyclic DLS of order N=2k+1 (added to OEIS as A341585):

**1, 0, 1, 1, 0, 2, 3, 0, 4, 4, 0, 5**

N≤13 — less than 1 second
N=17 — 1 minute (Core i7 4770, 1 thread)
N=19 — 1 hours 7 minutes
N=23 — 413 hours (2+ days on Core i7 4770, 8 threads)

N≥25 — parallel computing system required

Cyclic and pandiagonal properties of LS/DLS are not equivalent for rows and columns permuting. Is enumerating based on main classes possible? Open question…

## Brief conclusion

- new numerical series was calculated
- new upper and lower bounds for some series was established
- interconnections between Latin squares and different type combinatorial objects was established
- new method of Euler totient function calculating based on Latin squares was proposed

## Related work

Collecting CFs and new combinatorial structures search:

- triple of MODLS (is it exist?)
- different structures?

GPU implementation of transversal, cover and ESODLS algorithms?

Enumeration problems (OEIS):

- expanding current sequences
- enumerating DLS and ODLS of special kind (string-inverse, symmetric, …) and its CFs

Pseudo triples:

- 3 kinds of pseudo triples, only 1 was investigated in details

# Thank you for your attention!

Thanks to all the volunteers who took part in the Gerasim@home project!

WWW: http://evatutin.narod.ru, http://gerasim.boinc.ru
E-mail: evatutin@rambler.ru
LJ: http://evatutin.livejournal.com
Skype: evatutin