Southwest State University
Department of computer sciences

ON POLYNOMIAL REDUCTION OF PROBLEMS BASED ON DIAGONAL LATIN SQUARES TO THE EXACT COVER PROBLEM. AND RELATED RESULTS…

Vatutin E.I.

Kursk, 2019

What is Latin squares?

$$A = \left\| a_{ij} \right\|$$

$$i, j = \overline{1, N}$$

$$N = |S|$$

$$S = \{0, 1, 2, \ldots, N-1\}$$

$$\forall i, j, k = \overline{1, N}, j \neq k : (a_{ij} \neq a_{ik}) \wedge (a_{ji} \neq a_{ki})$$

$$\forall i, j = \overline{1, N}, i \neq j: (a_{ii} \neq a_{jj}) \wedge (a_{N-i+1, N-i+1} \neq a_{N-j+1, N-j+1})$$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 9 & 4 & 3 & 6 & 7 & 5 & 0 & 8 \\
2 & 9 & 3 & 1 & 7 & 0 & 5 & 8 & 4 & 6 \\
3 & 4 & 1 & 2 & 8 & 7 & 9 & 6 & 5 & 0 \\
4 & 3 & 5 & 9 & 2 & 1 & 8 & 0 & 6 & 7 \\
5 & 6 & 4 & 8 & 1 & 2 & 0 & 9 & 7 & 3 \\
6 & 5 & 8 & 7 & 0 & 3 & 2 & 1 & 9 & 4 \\
7 & 8 & 6 & 0 & 9 & 4 & 1 & 2 & 3 & 5 \\
8 & 7 & 0 & 5 & 6 & 9 & 3 & 4 & 1 & 2 \\
9 & 0 & 7 & 6 & 5 & 8 & 4 & 3 & 2 & 1
\end{pmatrix}$$

Normalized LS of order 10

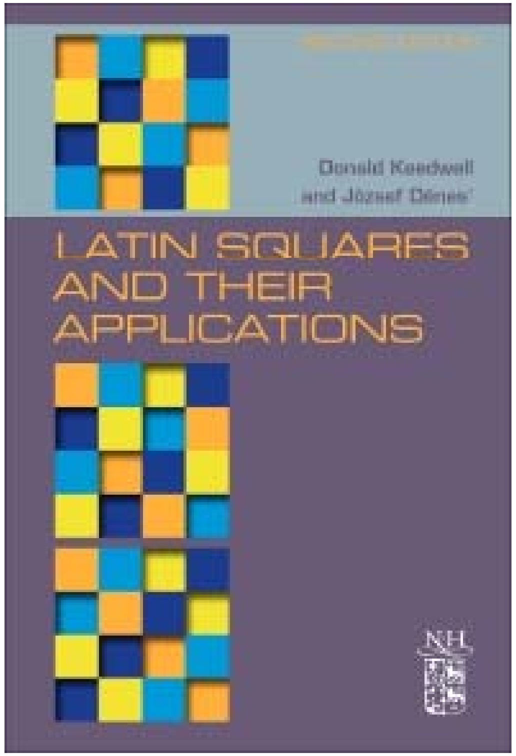
$$N! \times (N-1)!$$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
7 & 2 & 4 & 9 & 0 & 6 & 5 & 1 & 3 & 8 \\
8 & 3 & 6 & 7 & 5 & 9 & 0 & 2 & 4 & 1 \\
2 & 6 & 8 & 5 & 1 & 7 & 4 & 0 & 9 & 3 \\
5 & 8 & 9 & 1 & 7 & 0 & 3 & 4 & 6 & 2 \\
9 & 4 & 1 & 2 & 8 & 3 & 7 & 6 & 0 & 5 \\
4 & 7 & 5 & 6 & 9 & 1 & 8 & 3 & 2 & 0 \\
3 & 0 & 7 & 8 & 2 & 4 & 1 & 9 & 5 & 6 \\
6 & 5 & 0 & 4 & 3 & 2 & 9 & 8 & 1 & 7 \\
1 & 9 & 3 & 0 & 6 & 8 & 2 & 5 & 7 & 4
\end{pmatrix}$$

Normalized DLS of order 10

$$(N-1)!$$

Why is this interesting?



Applied problems:

- experiment planning
- cryptography
- error correcting codes
- scheduling
- algebra, combinatorics, statistics, ...

Mathematical problems:

- **existence of a triple of MOLS / MODLS**
- generating functions
- asymptotic behavior of combinatorial characteristics based on DLSs (OEIS)
- number theory (relations between different fields of knowledge)
- magic squares
- Sudoku (LS of order 9 with additional constraints)

Keedwell Donald, József Dénes. Latin Squares and their Applications (2nd Edition). Elsevier, 2015. 438 p.

Searching for pairs of ODLS of order 10

L. Euler expected that for N=10 ODLS doesn't exist

First pair — Parker et al., 1960

0	1	2	3	4	5	6	7	8	9
1	2	0	4	3	7	9	8	5	6
7	3	5	9	0	4	8	6	2	1
3	5	6	8	9	0	4	1	7	2
4	9	7	2	6	8	1	5	0	3
5	8	4	6	7	1	3	2	9	0
8	4	9	1	2	3	7	0	6	5
6	7	3	0	1	2	5	9	4	8
9	0	1	5	8	6	2	4	3	7
2	6	8	7	5	9	0	3	1	4

0	1	2	3	4	5	6	7	8	9
7	5	1	9	2	8	0	4	6	3
1	0	3	4	6	7	5	2	9	8
9	8	4	7	5	2	1	0	3	6
6	7	9	0	8	3	2	1	5	4
4	6	5	1	0	9	8	3	2	7
2	3	8	5	1	6	4	9	7	0
5	2	7	8	3	4	9	6	0	1
3	4	6	2	9	0	7	8	1	5
8	9	0	6	7	1	3	5	4	2

SAT@Home, 04.2015

0	1	2	3	4	5	6	7	8	9
4	9	0	8	5	6	3	1	2	7
2	5	7	9	6	4	0	8	1	3
9	0	4	6	8	7	1	5	3	2
6	7	5	2	1	3	8	0	9	4
1	8	3	5	7	2	9	6	4	0
7	3	1	0	9	8	4	2	6	5
8	2	6	4	0	9	5	3	7	1
3	4	8	1	2	0	7	9	5	6
5	6	9	7	3	1	2	4	0	8

0	1	2	3	4	5	6	7	8	9
6	5	9	7	0	8	2	3	1	4
4	7	1	2	3	9	8	0	6	5
1	2	0	4	5	3	7	6	9	8
2	6	8	0	9	4	1	5	3	7
8	4	6	9	2	7	0	1	5	3
5	0	4	6	8	2	3	9	7	1
9	3	5	1	7	6	4	8	0	2
7	8	3	5	6	1	9	4	2	0
3	9	7	8	1	0	5	2	4	6

Gerasim@Home, 04.2017

Very rare combinatorial objects:
~30 millions DLS of order 10
has only **1 pair of ODLS**!

Closest decision to the triple of MODLS

0	1	2	3	4	5	6	7	8	9
1	2	3	4	9	0	5	6	7	8
4	0	8	7	6	3	2	1	9	5
9	8	7	6	5	4	3	2	1	0
5	9	1	2	3	6	7	8	0	4
3	5	9	8	2	7	1	0	4	6
2	3	4	0	8	1	9	5	6	7
7	6	5	9	1	8	0	4	3	2
6	4	0	1	7	2	8	9	5	3
8	7	6	5	0	9	4	3	2	1

Orthogonality characteristic
74,
citerra
(**world record, 2016**)

0	1	2	3	4	5	6	7	8	9
9	8	7	6	5	4	3	2	1	0
5	0	6	8	7	2	1	3	9	4
1	6	4	7	9	0	2	5	3	8
4	9	3	1	2	7	8	6	0	5
8	3	5	2	0	9	7	4	6	1
3	7	0	4	8	1	5	9	2	6
7	4	8	9	6	3	0	1	5	2
2	5	1	0	3	6	9	8	4	7
6	2	9	5	1	8	4	0	7	3

Orthogonality characteristic
74,
evatutin (2017)

- Can characteristic value be increased? It is open question, we are trying...
- Are decisions differ?
- Are decisions have special properties?

Combinatorial structures for order 10

Strategies:

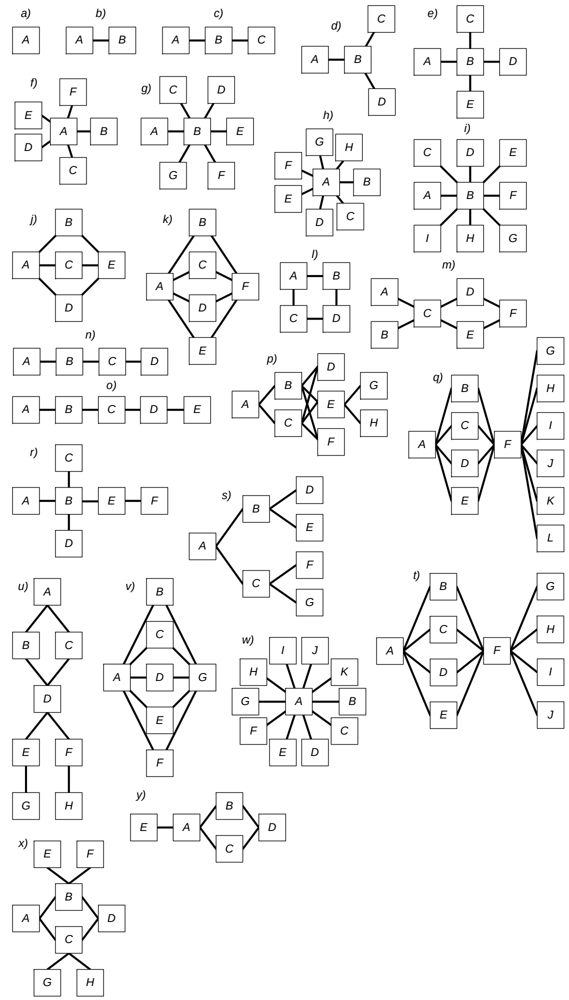
- direct search (ineffective);
- Euler-Parker method;
- special methods (rows rearrangement, SODLS, etc.).

Main schema:

Generator -> Processor -> Postprocessor

Bottleneck: <u>**processor**</u>!

New stage (from 04.2019): postprocessor!



Vatutin E.I., Titov V.S., Zaikin O.S., Kochemazov S.E., Manzuk M.O., Nikitina N.N. Orthogonality-based classification of diagonal Latin squares of order 10 // CEUR Workshop Proceedings. Vol. 2267. Proceedings of the VIII International Conference "Distributed Computing and Grid-technologies in Science and Education" (GRID 2018). Dubna, JINR, 2018. pp. 282–287.

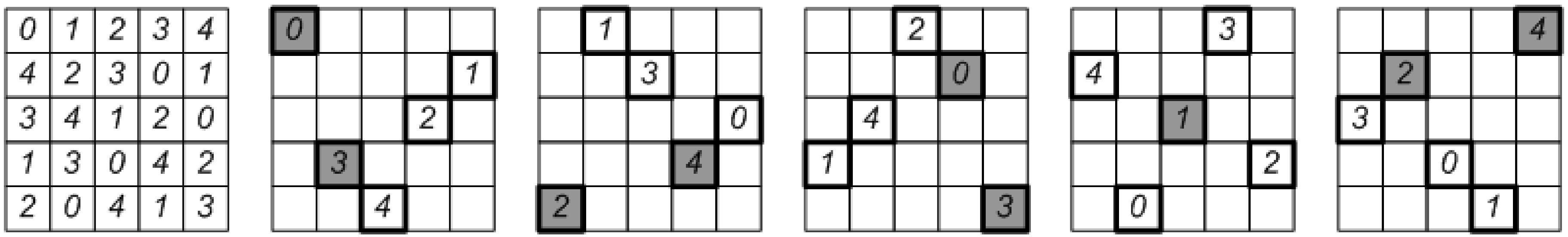
Euler-Parker based processor

Two stages:

- getting transversals set;
- getting subsets of N disjoint transversals.

Implementations:

- Brute Force (backtrack programming) — **~250-300 DLS/s**;
- Exact cover using (DLX) — **~600** (v1) ... **900** (v2) **DLS/s**;
- Canonizer using (A.D. Belyshev) — **~6000-8000 DLS/s** (inside).

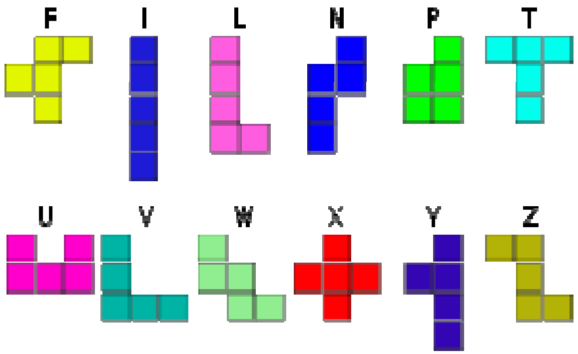


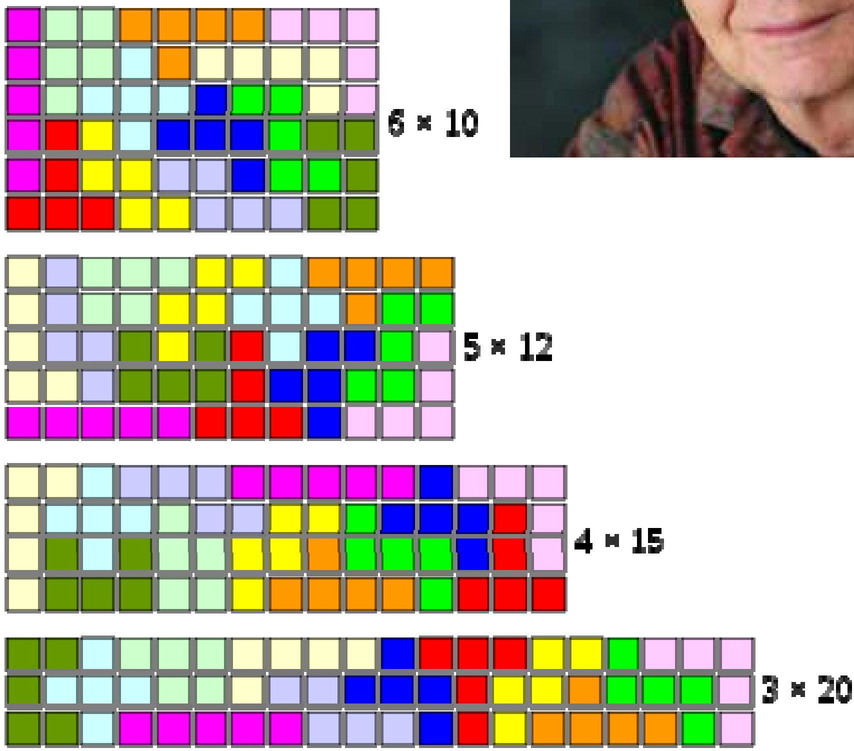
$T^{(d)}_1=\{a_{11}, a_{25}, a_{34}, a_{42}, a_{53}\}$ $T^{(d)}_2=\{a_{12}, a_{23}, a_{35}, a_{43}, a_{51}\}$ $T^{(d)}_3=\{a_{13}, a_{24}, a_{32}, a_{41}, a_{55}\}$ $T^{(d)}_4=\{a_{14}, a_{21}, a_{33}, a_{45}, a_{52}\}$ $T^{(d)}_5=\{a_{15}, a_{22}, a_{31}, a_{43}, a_{54}\}$

Dancing Links X algorithm (abbr. DLX)

Author: Donald E. Knuth, 2000

Problem: Exact Cover Problem (NP class)





https://habr.com/ru/post/194410/

Exact cover problem example

	Group 1: $r_i = v$									Group 2: $c_j = v$									Group 3: a_{ij} filled									Groups 4 and 5					
	$r_0 = 0$	$r_0 = 1$	$r_0 = 2$	$r_1 = 0$	$r_1 = 1$	$r_1 = 2$	$r_2 = 0$	$r_2 = 1$	$r_2 = 2$	$c_0 = 0$	$c_0 = 1$	$c_0 = 2$	$c_1 = 0$	$c_1 = 1$	$c_1 = 2$	$c_2 = 0$	$c_2 = 1$	$c_2 = 2$	a_{00}	a_{01}	a_{02}	a_{10}	a_{11}	a_{12}	a_{20}	a_{21}	a_{22}	$0 \in d_1$	$1 \in d_1$	$2 \in d_1$	$0 \in d_2$	$1 \in d_2$	$2 \in d_2$
$a_{00} := 0$	1									1									1									1					
$a_{00} := 1$		1									1				∈				1										1				
$a_{00} := 2$			1									1							1											1			
$a_{01} := 0$	1												1							1												
$a_{01} := 1$		1												1						1												
$a_{01} := 2$			1												1					1												
$a_{02} := 0$	1															1					1										1		
$a_{02} := 1$		1															1				1											1	
$a_{02} := 2$			1															1			1												1
$a_{10} := 0$				1						1												1											
$a_{10} := 1$					1						1											1											
$a_{10} := 2$						1						1										1											
$a_{11} := 0$				1									1										1					1			1		
$a_{11} := 1$					1									1									1						1			1	
$a_{11} := 2$						1									1								1							1			1
$a_{12} := 0$				1												1								1									
$a_{12} := 1$					1												1							1									
$a_{12} := 2$						1												1						1									
$a_{20} := 0$							1			1															1						1		
$a_{20} := 1$								1			1														1							1	
$a_{20} := 2$									1			1													1								1
$a_{21} := 0$							1						1													1							
$a_{21} := 1$								1						1												1							
$a_{21} := 2$									1						1											1							
$a_{22} := 0$							1									1											1	1					
$a_{22} := 1$								1									1										1		1				
$a_{22} := 2$									1									1									1			1			

Cover matrix for DLSs of order 3 generation

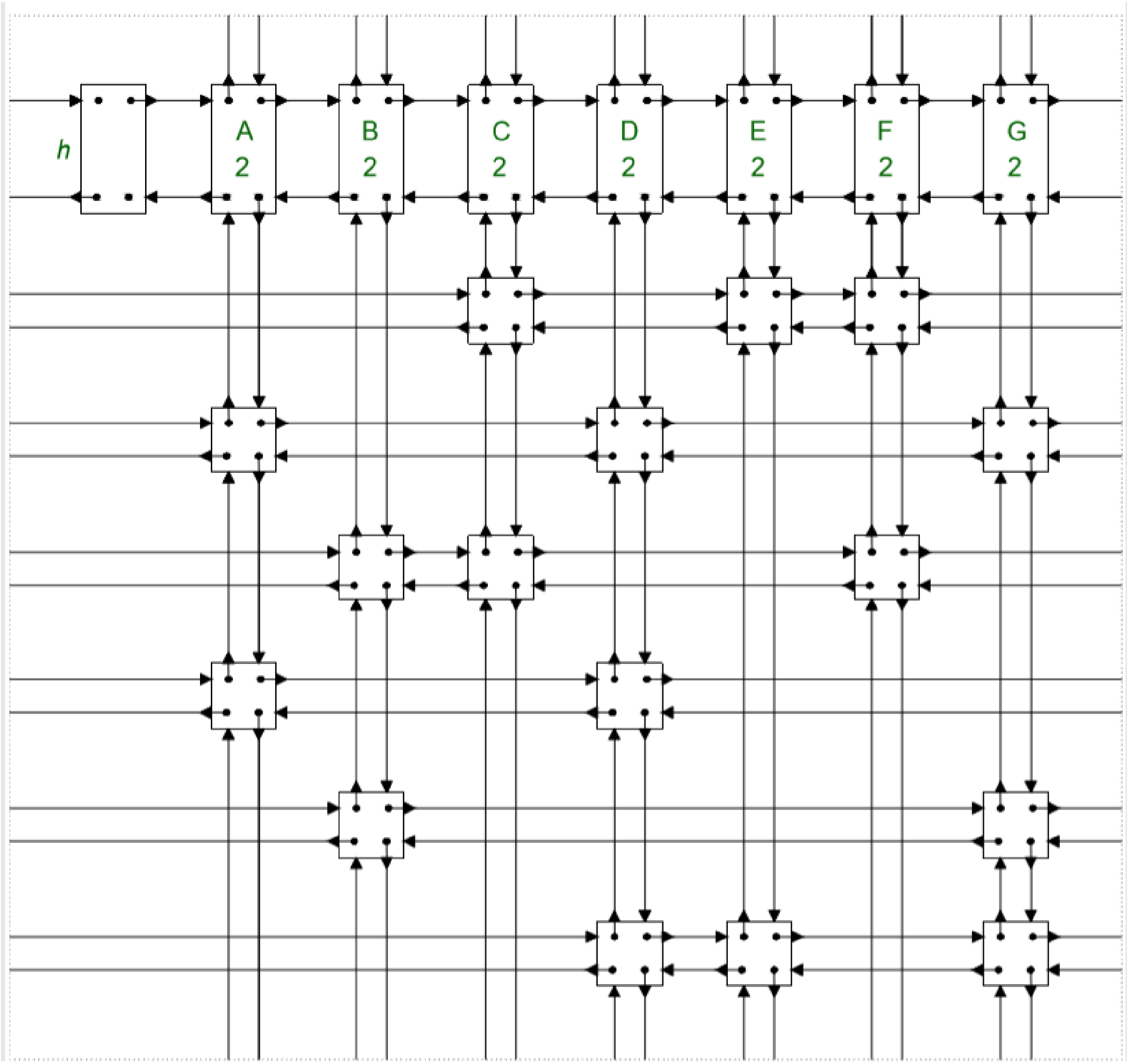
Latin squares problems -> Exact cover problem

Latin squares connected problems:

- LS/DLS generation;
- special types of LS/DLS generation (normalized, symmetric, etc.);
- getting transversals and diagonal transversals;
- getting disjoint sets of transversals;
- direct search of ODLSs for given DLS.

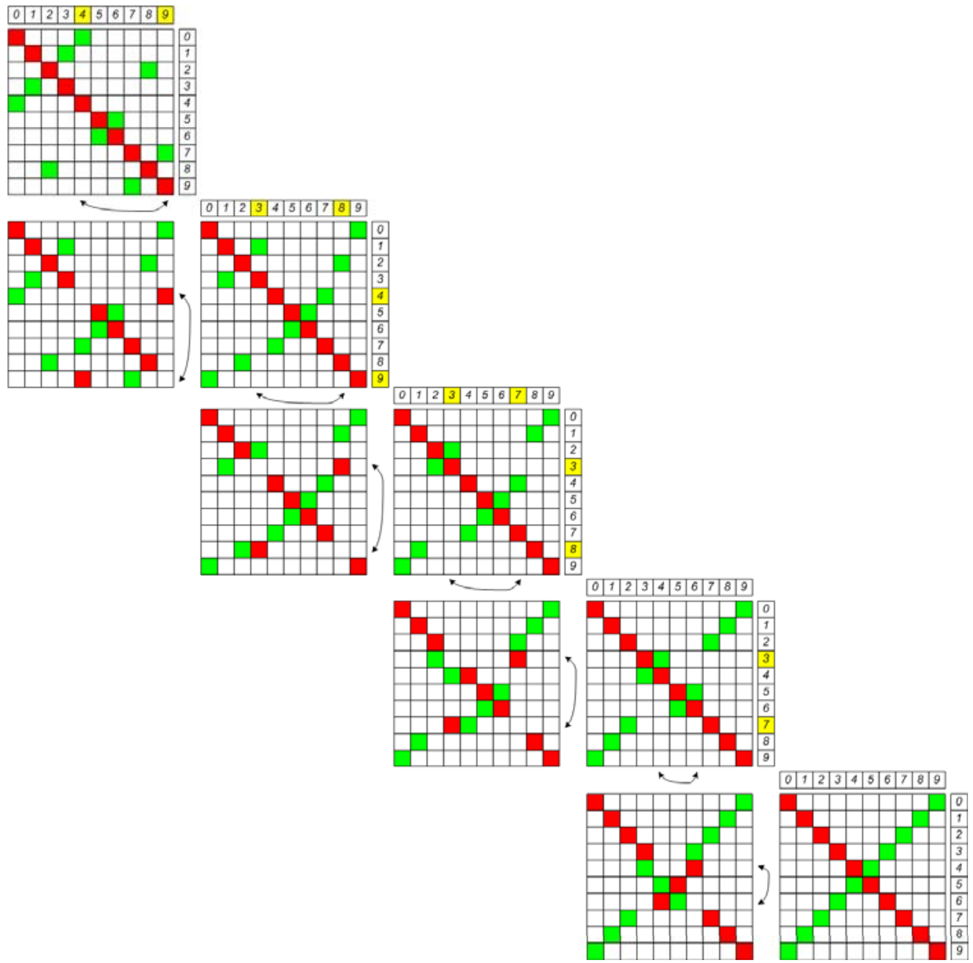
Dancing links

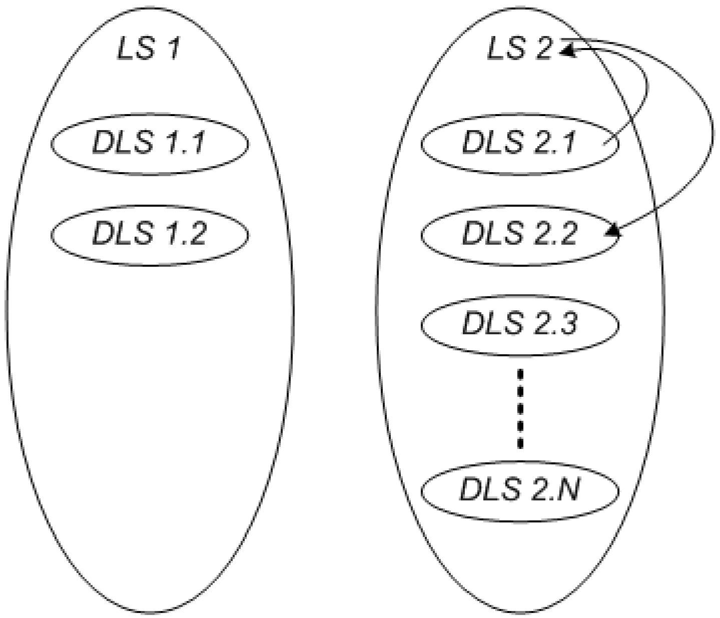
Fast implementation of cover/uncover operations



Canonization

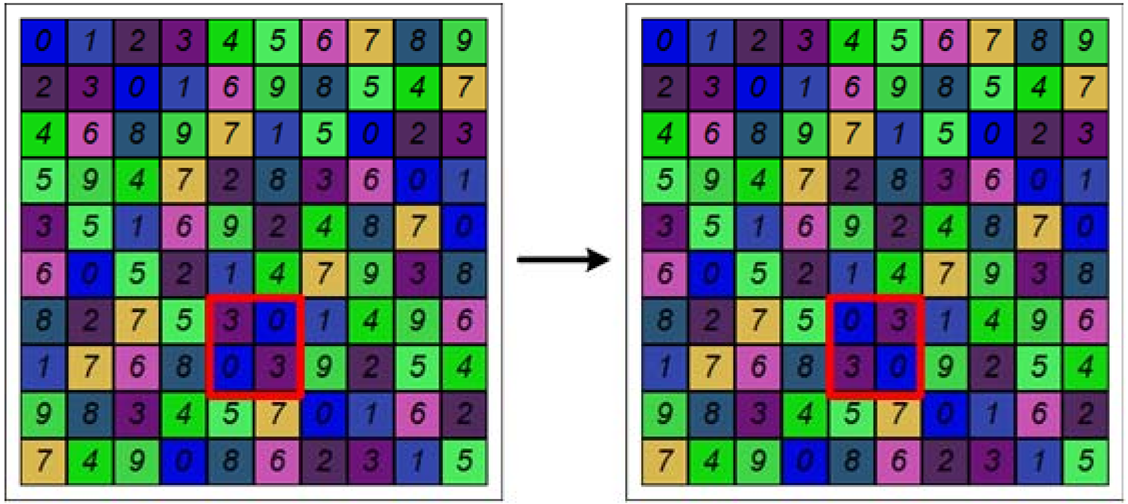
Main idea (Brown et al.): find pair of the symmetrically placed transversals, permute rows and columns of LS to get DLS.





Simple transformations

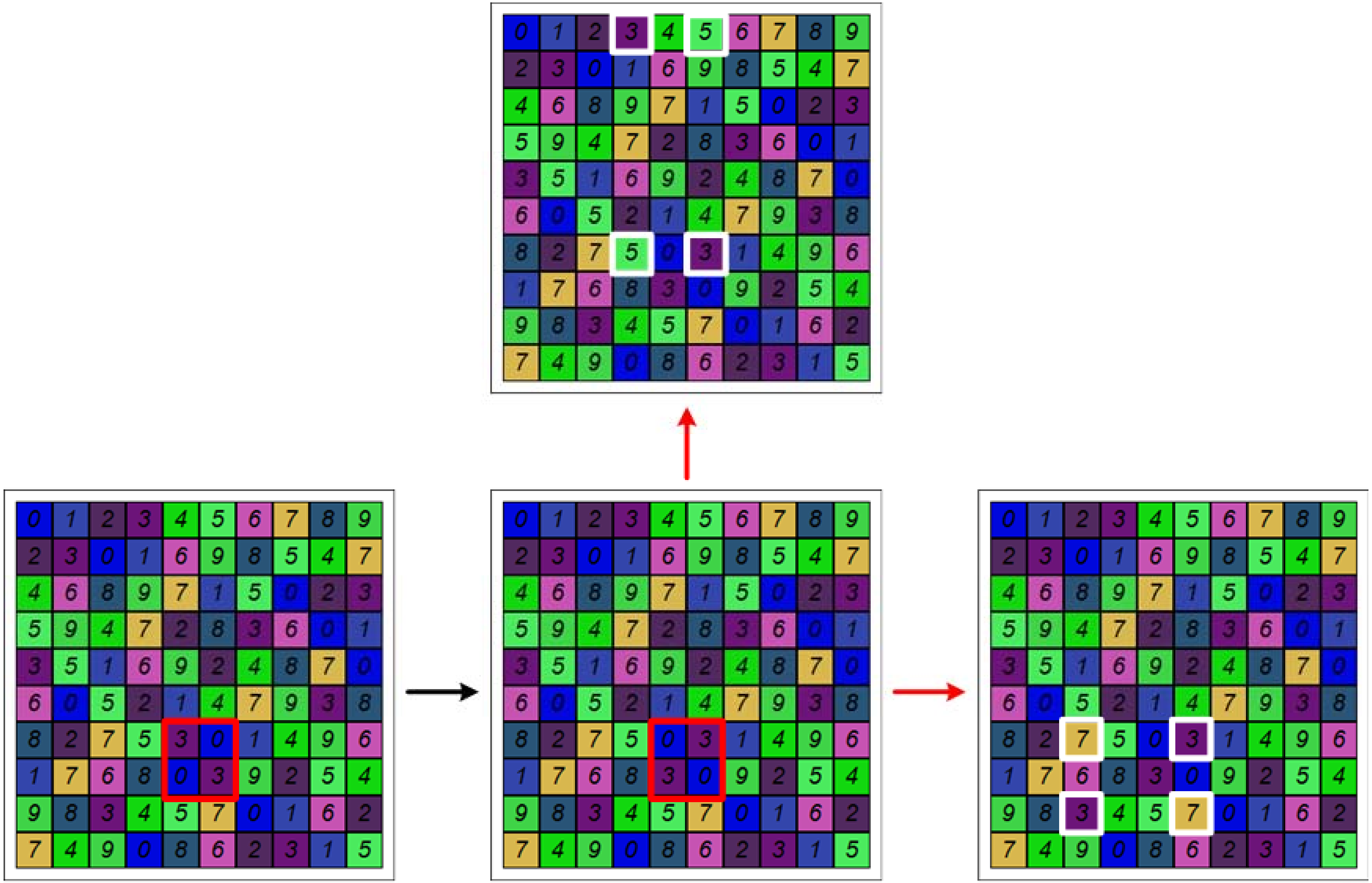
What is intercalate?



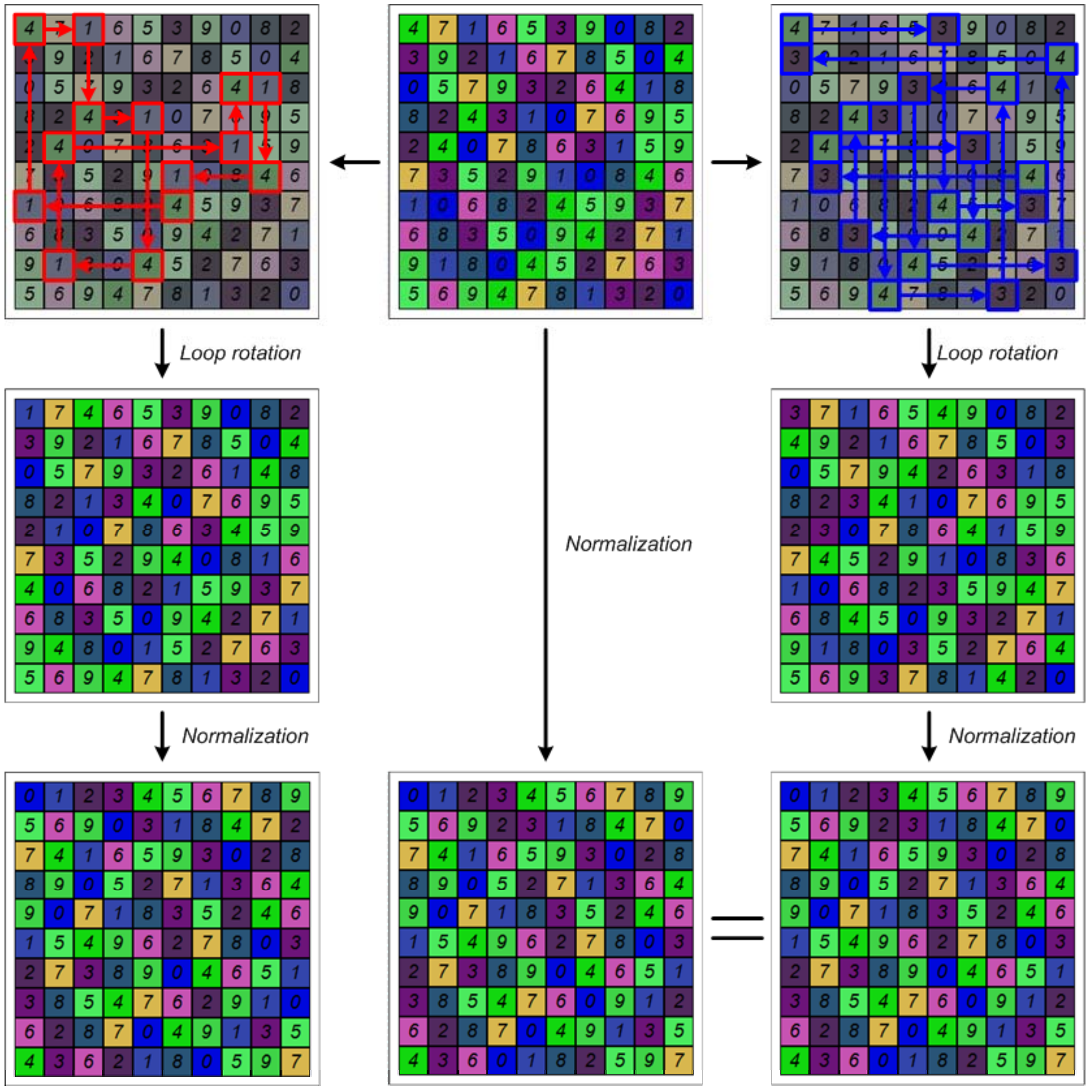
Latin square of size 2x2:

- columns i1 and i2;
- rows j1 and j2.

What is nested intercalates?



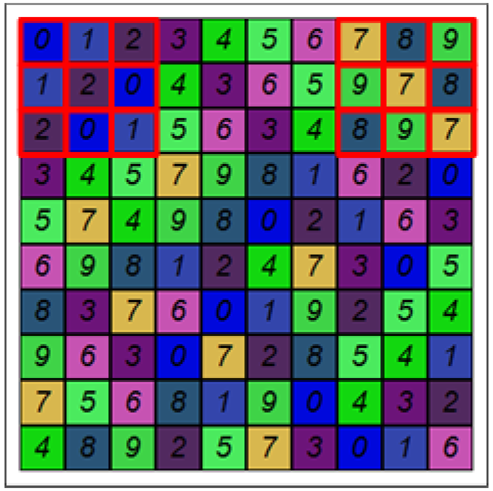
What is loop?



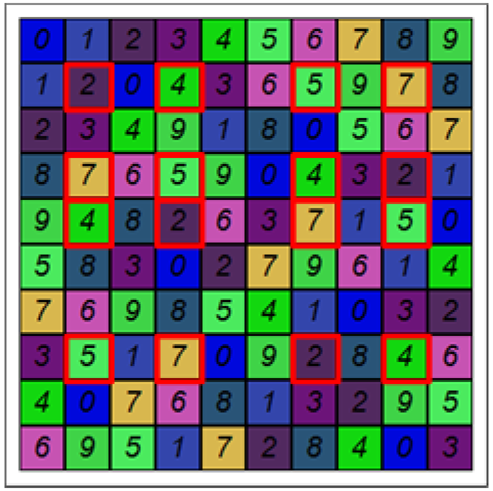
- full and short loops;
- shortest possible loop is intercalate.

What is Latin subrectangles and Latin subsquares?

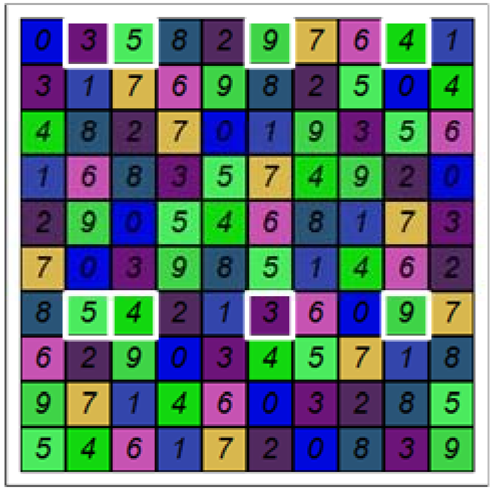
Once, 2 subLS 3x3



Line-5, subLS 4x4



1:2 structure, subLR 2x4



- trivial and nontrivial subLRs;
- smallest nontrivial subLR is intercalate.

Strategies of simple transformations

Stage 1 (one of):

- rotate K intercalates;
- rotate K short loops;
- «rotate» K nontrivial subLRs.

Remark: after transformation some squares can be correct LSs, but not DLSs!

Stage 2 (one of):

- Euler-Parker method + DLX (processing for DLSs only, fast);
- Canonizer (processing for LSs, slow, but...).

What strategy is preferable? Experiment:

- average ODLS CF «cost» without postprocessing — **<u>3000 s</u>** (8,3 h);
- average ODLS CF «cost» after different simple transformations closure — **<u>4 s</u>** ... **<u>16 000 000 s</u>**.

3	2	8	4	6	7	1	0	9	5
8	1	2	7	4	6	3	5	0	9
1	5	0	9	8	2	4	3	7	6
6	8	5	2	0	9	7	1	4	3
9	0	7	1	5	4	2	6	3	8
4	3	9	0	1	8	6	7	5	2
0	6	3	8	7	5	9	2	1	4
5	7	6	3	9	1	8	4	2	0
7	9	4	5	2	3	0	8	6	1
2	4	1	6	3	0	5	9	8	7

↓

3	2	8	4	6	7	1	0	9	5
8	1	2	7	4	6	3	5	0	9
1	5	0	9	8	2	4	3	7	6
2	8	5	6	0	9	7	1	4	3
9	0	7	1	5	4	2	6	3	8
4	3	9	0	1	8	6	7	5	2
0	6	3	8	7	5	9	2	1	4
5	7	6	3	9	1	8	4	2	0
7	9	4	5	2	3	0	8	6	1
6	4	1	2	3	0	5	9	8	7

Simple transformations: results

- **+10-15%** more ODLS CFs with postprocessing;
- **+72** loop-4 CFs, **+8** line-4 CFs and **+4** 1:3 CFs after postprocessing (very rare combinatorial structures!).

Summary

Some results...

Generalized symmetries exploration

Обработаны, число находок (двушек)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
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Recognition — 2018

Обработаны, число находок (двушек)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
1	-	24	31	22	41	26	21	32/25	18	31	23	25	17	20	25	1357/17	19	19	19	20	22	22	43/35	24	22	21	10	19	14	15	2386	62	18	18	15	12	12	27	20	20	###	14
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4												25				44/25			23				21								2995											18
5												18				31/28			18				30								365/33											
6												17				24/23			27				11								36											
7							17					24								14		13									38/17										16	16
8								29				16									23		22								20											15
9													13	21	11					16		22									23											
10										17									16				14								25/23			12								
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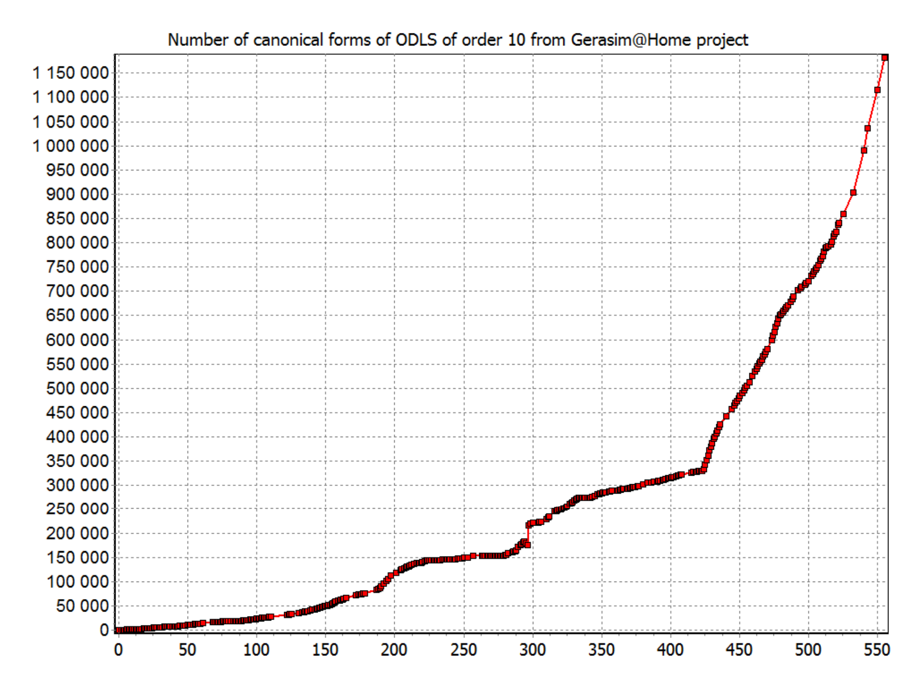
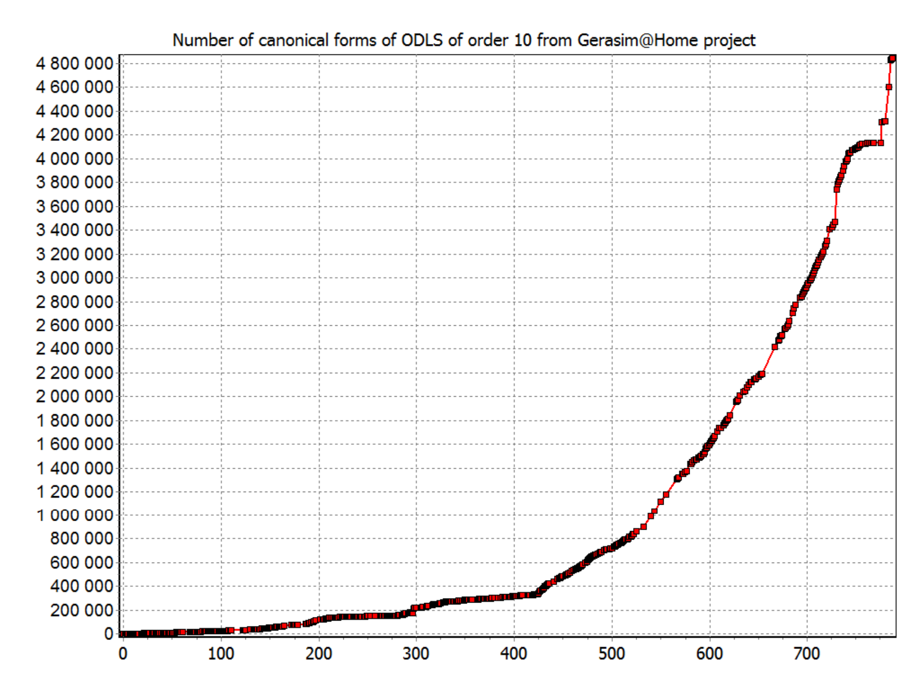
Recognition — 2019

Getting ODLS CFs within Gerasim@Home project

Strategy of search: getting source square (random generator, symmetric random generator), try to get orthogonal square, add the unique CF to collection



Recognition — 2018
(1 100 000+ CFs)



Recognition — 2019
(4 800 000+ CFs,
4,3x)

Online Encyclopedia of Integer Sequences

Main classes of DLS:

- A287764 — Number of main classes of diagonal Latin squares of order N (N<9)
- A299783 — Minimal size of main class for diagonal Latin squares of order N with fixed first row (N<9)
- A299784 — Maximal size of main class for diagonal Latin squares of order N with fixed first row (N<9)
- A299785 — Minimal size of main class for diagonal Latin squares of order N (N<9)
- A299787 — Maximal size of main class for diagonal Latin squares of order N (N<9)

Intercalates, loops, Latin subrectangles and subsquares in DLS:

- A307163 — Minimum number of intercalates in a diagonal Latin square of order N (N<9)
- A307164 — Maximum number of intercalates in a diagonal Latin square of order N (N<9)
- A307166 — Minimum number of loops in a diagonal Latin square of order N (N<8)
- A307167 — Maximum number of loops in a diagonal Latin square of order N (N<8)
- A307170 — Minimum number of partial loops in a diagonal Latin square of order N (N<8)
- A307171 — Maximum number of partial loops in a diagonal Latin square of order N (N<8)
- A307839 — Minimum number of Latin subrectangles in a diagonal Latin square of order N (N<8)
- A307840 — Maximum number of Latin subrectangles in a diagonal Latin square of order N (N<8)
- A307841 — Minimum number of nontrivial Latin subrectangles in a diagonal Latin square of order N (N<8)
- A307842 — Maximum number of nontrivial Latin subrectangles in a diagonal Latin square of order N (N<8)

- https://oeis.org

Online Encyclopedia of Integer Sequences: example of the numerical series A307841

A307841 Minimum number of nontrivial Latin subrectangles in a diagonal Latin square of order n. 0

0, 0, 0, 12, 0, 51, 0 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,4

COMMENTS A Latin subrectangle is an m X k Latin rectangle of a Latin square of order n, 1 <= m <= n, 1 <= k <= n.

A nontrivial Latin subrectangle is an m X k Latin rectangle of a Latin square of order n, 1 < m < n, 1 < k < n.

LINKS Table of n, a(n) for n=1..7.

E. I. Vatutin, Discussion about properties of diagonal Latin squares at forum.boinc.ru (in Russian)

Index entries for sequences related to Latin squares and rectangles

EXAMPLE For example, the square

```
0 1 2 3 4 5 6
4 2 6 5 0 1 3
3 6 1 0 5 2 4
6 3 5 4 1 0 2
1 5 3 2 6 4 0
5 0 4 6 2 3 1
2 4 0 1 3 6 5
```

has a nontrivial Latin subrectangle

```
. . . . . . .
. . 6 5 0 1 3
. . . . . . .
. . . . . . .
. . . . . . .
. . . . . . .
. . 0 1 3 6 5
```

The total number of Latin subrectangles for this square is 2119 and the number of nontrivial Latin subrectangles is only 151.

CROSSREFS Cf. A307839, A307842.

Sequence in context: A307170 A225951 A278711 * A257949 A077351 A119530

Adjacent sequences: A307838 A307839 A307840 * A307842 A307844 A307845

KEYWORD nonn,more,new

AUTHOR Eduard I. Vatutin, May 01 2019

STATUS approved

- https://oeis.org

Combinatorial structures of order 1-9

90. Структура 30N87M30C

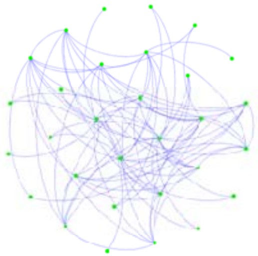


Figure 16. Graph of ODLS from workunit R9_000153990 1.
2017.10.23. RakeSearch project. Rally (BOINC Confederation) and amdigarek (SETI.Germany)

Рис. Комбинаторная структура

ДЛК, входящие в состав комбинаторной структуры:
- ДЛК 1: 0123456781204568375380724616817205437648130253075641828562073144 7 368125024513 8706;
- ДЛК 2: 0123456783417285066875301421348520672051647335604733214736812508 5 62073147280164 35;
- …
- ДЛК 30: 0123456782805371466051247835374028613417685027680134254736812508 562703141248560 37.

Множество различных КФ в составе комбинаторной структуры:
- КФ 1: 0123456781204568375380724616817205437648130253075641828562073144 73 68125024513 8706;
- КФ 2: 0123456781204783568752031645861374027348605216580142374635827103 47 621085201756843;
- …
- КФ 30: 0123456781205638474356017825487203616872341057631580248760124533 5 487621020148753 6.

Отсортированный вектор степеней вершин:
1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 10, 10, 10, 12, 14, 14.

Способ нахождения:
проект RakeSearch (строчно-перестановочные ОДЛК) + метод Эйлера-Паркера + DLX.

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91. Структура 30N98M15C

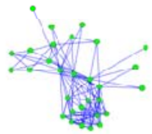

Figure 110. Graph of ODLS from workunit R9_000155414 01.
2017.12.03. Rake Search project. Bryan (SETI.USA) and PDW (0UK - Overclockers UK)

Рис. Комбинаторная структура

ДЛК, входящие в состав комбинаторной структуры:
- ДЛК 1: 0123456781234580678460725132375814065016347826842071357601238543 7 5816240458760321;
- ДЛК 2: 0123456786341287105375816240450723861128460357763158024287501436 84 607251350163478 2,
- …
- ДЛК 30: 0123456786382714053758162401504238678271603544637085212810547367 4658201350463718 2.

Множество различных КФ в составе комбинаторной структуры:
- КФ 1: 0123456781234580678460725132375814065016347826842071357601238543 75 81624045876032 1;
- КФ 2: 0123456781204867354037581626718305245682740137345612808561234073 47 602851285017346;
- …
- КФ 15: 0123456781230768453641087524756812302078534618512673047805341265 3 641208764872051 3.

Отсортированный вектор степеней вершин:
1, 1, 2, 2, 2, 2, 4, 4, 4, 4, 4, 4, 6, 6, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 12, 12, 15, 15.

Способ нахождения:
проект RakeSearch (строчно-перестановочные ОДЛК) + метод Эйлера-Паркера
DLX.

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- RakeSearch & Gerasim@Home results;
- http://evatutin.narod.ru/evatutin_ls_all_structs_n1to8_eng.pdf;
- http://evatutin.narod.ru/evatutin_ls_all_structs_n9_rus.pdf.

I have some additional minutes? :)

Related works...

Related work

Collecting CFs and new combinatorial structures search:

- triple of MODLS (is it exist?)
- different structures? (including different orders)

GPU implementation of transversal and cover algorithms?

Enumeration problems (OEIS):

- expanding current sequences
- enumerating DLS and ODLS of special kind (row-inverse, symmetric, ...) and its CFs

Pseudo triples:

- 3 kinds of pseudo triples, only 1 was investigated in details

Thank you for your attention!

Thanks to all the volunteers who took part in the Gerasim@home project!

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E-mail: evatutin@rambler.ru
LJ: http://evatutin.livejournal.com
Skype: evatutin