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Modification of the load balancing method in a desktop grid for solving problems of constructing Latin square spectra

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LS and DLS definition

$$A = \left\| a_{ij} \right\|$$

$$i,\ j = \overline{1,\ N}$$

$$N = \left| S \right|$$

$$S = \{0, 1, 2, ..., N-1\}$$

$$\forall i,\ j, k = \overline{1,\ N},\ j \neq k : \left(a_{ij} \neq a_{ik}\right) \wedge \left(a_{ji} \neq a_{ki}\right)$$

$$\forall i,\ j = \overline{1,\ N}, i \neq j : \left(a_{ii} \neq a_{jj}\right) \wedge \left(a_{N-i+1,\ N-i+1} \neq a_{N-j+1,\ N-j+1}\right)$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 9 & 4 & 3 & 6 & 7 & 5 & 0 & 8 \\ 2 & 9 & 3 & 1 & 7 & 0 & 5 & 8 & 4 & 6 \\ 3 & 4 & 1 & 2 & 8 & 7 & 9 & 6 & 5 & 0 \\ 4 & 3 & 5 & 9 & 2 & 1 & 8 & 0 & 6 & 7 \\ 5 & 6 & 4 & 8 & 1 & 2 & 0 & 9 & 7 & 3 \\ 6 & 5 & 8 & 7 & 0 & 3 & 2 & 1 & 9 & 4 \\ 7 & 8 & 6 & 0 & 9 & 4 & 1 & 2 & 3 & 5 \\ 8 & 7 & 0 & 5 & 6 & 9 & 3 & 4 & 1 & 2 \\ 9 & 0 & 7 & 6 & 5 & 8 & 4 & 3 & 2 & 1 \end{pmatrix}$$

Normalized LS of order 10

$$N! \times (N-1)!$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 2 & 4 & 9 & 0 & 6 & 5 & 1 & 3 & 8 \\ 8 & 3 & 6 & 7 & 5 & 9 & 0 & 2 & 4 & 1 \\ 2 & 6 & 8 & 5 & 1 & 7 & 4 & 0 & 9 & 3 \\ 5 & 8 & 9 & 1 & 7 & 0 & 3 & 4 & 6 & 2 \\ 9 & 4 & 1 & 2 & 8 & 3 & 7 & 6 & 0 & 5 \\ 4 & 7 & 5 & 6 & 9 & 1 & 8 & 3 & 2 & 0 \\ 3 & 0 & 7 & 8 & 2 & 4 & 1 & 9 & 5 & 6 \\ 6 & 5 & 0 & 4 & 3 & 2 & 9 & 8 & 1 & 7 \\ 1 & 9 & 3 & 0 & 6 & 8 & 2 & 5 & 7 & 4 \end{pmatrix}$$

Normalized DLS of order 10

$$(N-1)!$$

Numerical characteristics of LS/DLS

Fast computable:

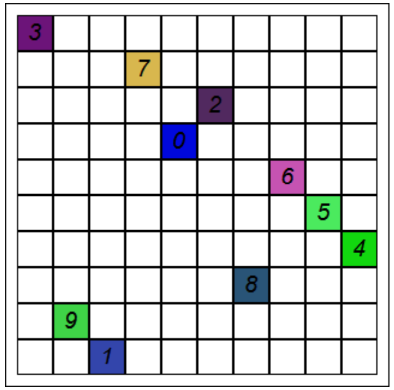
- intercalates number;
- transversals (general type or diagonal) number;

Relatively fast to compute:

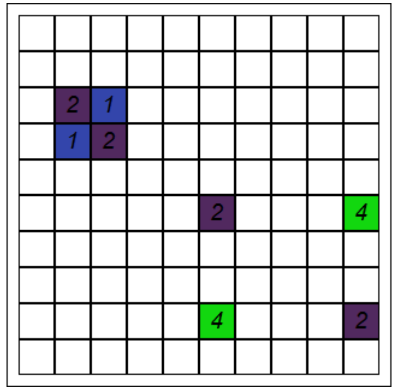
- OLS/ODLS number.

Computable intensive:

- full/short loops number;
- trivial/nontrivial Latin subrectangles number;
- main class cardinality;
- …



Transversal example



Intercalates example

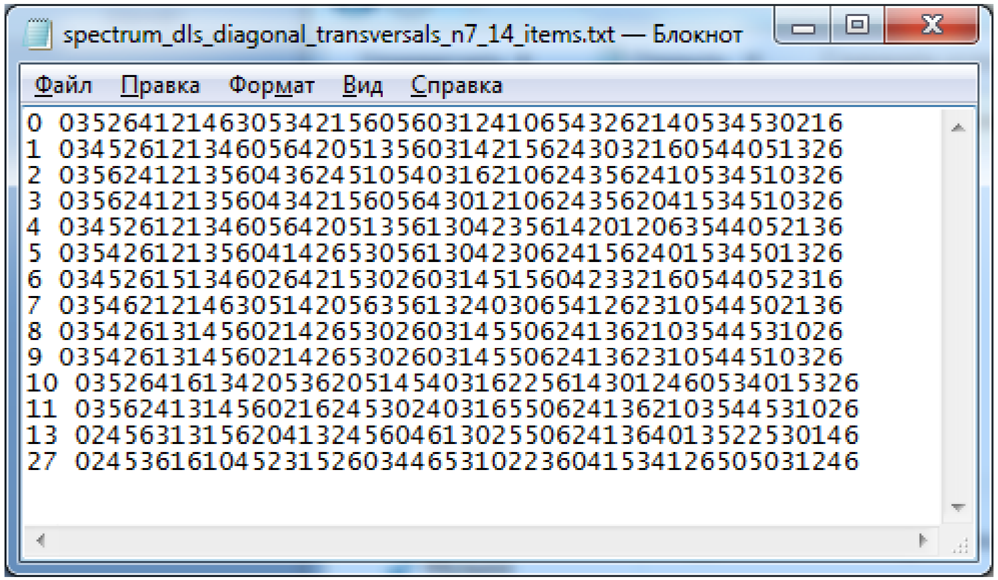
Numerical characteristic spectrum definition

Spectrum S — a set of numerical values of a selected numerical characteristic X of combinatorial objects of a given type.

Example:

- type of combinatorial objects — DLS of order 7;
- numerical characteristic — diagonal transversals number.

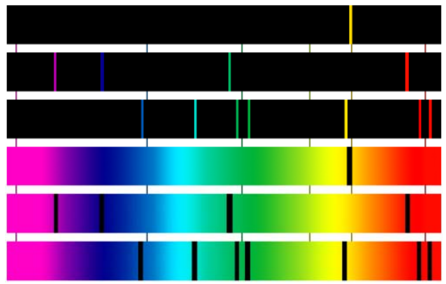
$$S_{dt,\,DLS\,7} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 27\}$$

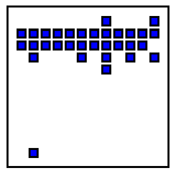


spectrum_dls_diagonal_transversals_n7_14_items.txt — Блокнот

Файл Правка Формат Вид Справка

```
0  0352641214630534215605603124106543262140534530216
1  0345261213460564205135603142156243032160544051326
2  0356241213560436245105403162106243562410534510326
3  0356241213560434215605643012106243562041534510326
4  0345261213460564205135613042356142012063544052136
5  0354261213560414265305613042306241562401534501326
6  0345261513460264215302603145156042332160544052316
7  0354621214630514205635613240306541262310544502136
8  0354261314560214265302603145506241362103544531026
9  0354261314560214265302603145506241362310544510326
10  0352641613420536205145403162256143012460534015326
11  0356241314560216245302403165506241362103544531026
13  0245631315620413245604613025506241364013522530146
27  0245361610452315260344653102236041534126505031246
```





http://evatutin.narod.ru/spectra/spectrum_dls_diagonal_transversals_n7_14_items.txt

Numerical series in OEIS

A345760 a(n) is the number of distinct numbers of intercalates of order n diagonal Latin squares.

0, 0, 0, 1, 2, 1, 21, 61, 64
(list; graph; refs; listen; history; text; internal format)

OFFSET 1,5

COMMENTS a(n) <= A307164(n) - A307163(n) + 1.
a(n) <= A287764(n).
a(10) >= 98, a(11) >= 145, a(12) >= 259, a(13) >= 200, a(14) >= 362, a(15) >= 536, a(16) >= 792, a(17) >= 685, a(18) >= 535, a(19) >= 447, a(20) >= 1011, a(21) >= 747, a(22) >= 872, a(23) >= 885, a(24) >= 1610, a(25) >= 1677, a(26) >= 1266, a(27) >= 1337, a(28) >= 2795. - Eduard I. Vatutin, Oct 02 2021, updated Mar 02 2025

LINKS Table of n, a(n) for n=1..9.
Eduard I. Vatutin, About the spectra of numerical characteristics of diagonal Latin squares of orders 1-7 (in Russian).
Eduard I. Vatutin, About the spectra of numerical characteristics of diagonal Latin squares of order 8 (in Russian).
Eduard I. Vatutin, About the approximation of spectra of numerical characteristics of diagonal Latin squares of order 9 (in Russian).
Eduard I. Vatutin, About the results of experiment with spectra of diagonal Latin squares using Brute Force and distributed computing projects Gerasim@Home and RakeSearch (in Russian).
Eduard I. Vatutin, Graphical representation of the spectra.
Eduard I. Vatutin, Proving lists (1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26).
E. I. Vatutin, N. N. Nikitina, M. O. Manzuk, I. I. Kurochkin, A. M. Albertyan, A. V. Kripachev, A. I. Pykhtin, Methods for getting spectra of fast computable numerical characteristics of diagonal Latin squares, Cloud and distributed computing systems in electronic control conference, within the National supercomputing forum (NSCF - 2022). Pereslavl-Zalessky, 2023. pp. 19-23. (in Russian)
E. I. Vatutin, V. S. Titov, A. I. Pykhtin, A. V. Kripachev, N. N. Nikitina, M. O. Manzuk, A. M. Albertyan and I. I. Kurochkin, Estimation of the Cardinalities of the Spectra of Fast-computable Numerical Characteristics for Diagonal Latin Squares of Orders N>9 (in Russian) // Science and education in the development of industrial, social and economic spheres of Russian regions. Murom, 2022. pp. 314-315.
E. I. Vatutin, V. S. Titov, A. I. Pykhtin, A. V. Kripachev, N. N. Nikitina, M. O. Manzuk, A. M. Albertyan, I. I. Kurochkin, Heuristic method for getting approximations of spectra of numerical characteristics for diagonal Latin squares, Intellectual information systems: trends, problems, prospects, Kursk, 2022. pp. 35-41. (in Russian)
Index entries for sequences related to Latin squares and rectangles.

EXAMPLE For n=7 the number of intercalates that a diagonal Latin square of order 7 may have is 0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 22, 26, or 30. Since there are 21 distinct values, a(7)=21.

CROSSREFS Cf. A287764, A307163, A307164, A309344, A344105, A345370, A345761.
Sequence in context: A051492 A380220 A164827 * A213976 A330354 A200859
Adjacent sequences: A345757 A345758 A345759 * A345761 A345762 A345763

KEYWORD nonn,more,hard

AUTHOR Eduard I. Vatutin, Jun 26 2021

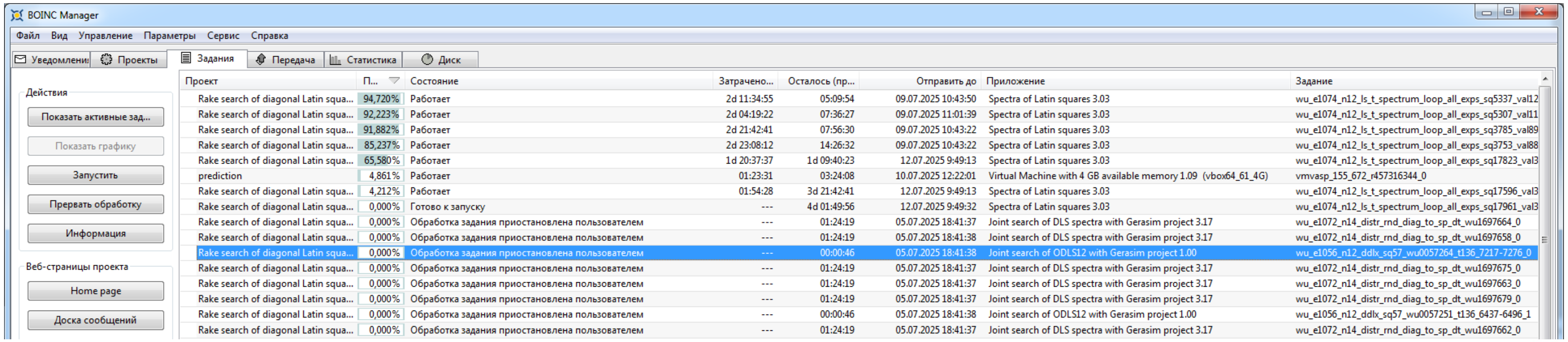
EXTENSIONS a(9) added by Eduard I. Vatutin, Oct 22 2022

STATUS approved

- https://oeis.org
- more information: https://youtu.be/rAWfBZqwCl4

Current experiments set

- spectrum of ODLS number for DLS of order 12;
  - low part – 1 threaded software implementation, **active** (RakeSearch project in future)
  - high part – RakeSearch project (experiment number e1056, **active**)
- filling the collection of ODLS of order 10 – RakeSearch project (experiment number e43, **active**)
- distributed random diagonalizing of spectrum of diagonal transversals number of DLS of order 14 – RakeSearch project (experiment number e1072, **active**)
- expanding spectrum of transversals number in LS of order 12 using M2 method – RakeSearch project (experiment number e1074, **tail computation** (3-12 days per WU))
- expanding spectra of intercalates number in LS/DLS of orders 24-… RakeSearch project (experiments numbers e1075-e1078, **waiting for results of the previous experiment**)



Проект	П...	Состояние	Затрачено...	Осталось (пр...	Отправить до	Приложение	Задание
Rake search of diagonal Latin squa...	94,720%	Работает	2d 11:34:55	05:09:54	09.07.2025 10:43:50	Spectra of Latin squares 3.03	wu_e1074_n12_ls_t_spectrum_loop_all_exps_sq5337_val12
Rake search of diagonal Latin squa...	92,223%	Работает	2d 04:19:22	07:36:27	09.07.2025 11:01:39	Spectra of Latin squares 3.03	wu_e1074_n12_ls_t_spectrum_loop_all_exps_sq5307_val11
Rake search of diagonal Latin squa...	91,882%	Работает	2d 21:42:41	07:56:30	09.07.2025 10:43:22	Spectra of Latin squares 3.03	wu_e1074_n12_ls_t_spectrum_loop_all_exps_sq3785_val89
Rake search of diagonal Latin squa...	85,237%	Работает	2d 23:08:12	14:26:32	09.07.2025 10:43:22	Spectra of Latin squares 3.03	wu_e1074_n12_ls_t_spectrum_loop_all_exps_sq3753_val88
Rake search of diagonal Latin squa...	65,580%	Работает	1d 20:37:37	1d 09:40:23	12.07.2025 9:49:13	Spectra of Latin squares 3.03	wu_e1074_n12_ls_t_spectrum_loop_all_exps_sq17823_val3
prediction	4,861%	Работает	01:23:31	03:24:08	10.07.2025 12:22:01	Virtual Machine with 4 GB available memory 1.09 (vbox64_61_4G)	vmvasp_155_672_r457316344_0
Rake search of diagonal Latin squa...	4,212%	Работает	01:54:28	3d 21:42:41	12.07.2025 9:49:13	Spectra of Latin squares 3.03	wu_e1074_n12_ls_t_spectrum_loop_all_exps_sq17596_val3
Rake search of diagonal Latin squa...	0,000%	Готово к запуску	---	4d 01:49:56	12.07.2025 9:49:32	Spectra of Latin squares 3.03	wu_e1074_n12_ls_t_spectrum_loop_all_exps_sq17961_val3
Rake search of diagonal Latin squa...	0,000%	Обработка задания приостановлена пользователем	---	01:24:19	05.07.2025 18:41:37	Joint search of DLS spectra with Gerasim project 3.17	wu_e1072_n14_distr_rnd_diag_to_sp_dt_wu1697664_0
Rake search of diagonal Latin squa...	0,000%	Обработка задания приостановлена пользователем	---	01:24:19	05.07.2025 18:41:38	Joint search of DLS spectra with Gerasim project 3.17	wu_e1072_n14_distr_rnd_diag_to_sp_dt_wu1697658_0
Rake search of diagonal Latin squa...	0,000%	Обработка задания приостановлена пользователем	---	00:00:46	05.07.2025 18:41:38	Joint search of ODLS12 with Gerasim project 1.00	wu_e1056_n12_ddlx_sq57_wu0057264_t136_7217-7276_0
Rake search of diagonal Latin squa...	0,000%	Обработка задания приостановлена пользователем	---	01:24:19	05.07.2025 18:41:37	Joint search of DLS spectra with Gerasim project 3.17	wu_e1072_n14_distr_rnd_diag_to_sp_dt_wu1697675_0
Rake search of diagonal Latin squa...	0,000%	Обработка задания приостановлена пользователем	---	01:24:19	05.07.2025 18:41:37	Joint search of DLS spectra with Gerasim project 3.17	wu_e1072_n14_distr_rnd_diag_to_sp_dt_wu1697663_0
Rake search of diagonal Latin squa...	0,000%	Обработка задания приостановлена пользователем	---	01:24:19	05.07.2025 18:41:37	Joint search of DLS spectra with Gerasim project 3.17	wu_e1072_n14_distr_rnd_diag_to_sp_dt_wu1697679_0
Rake search of diagonal Latin squa...	0,000%	Обработка задания приостановлена пользователем	---	00:00:46	05.07.2025 18:41:38	Joint search of ODLS12 with Gerasim project 1.00	wu_e1056_n12_ddlx_sq57_wu0057251_t136_6437-6496_1
Rake search of diagonal Latin squa...	0,000%	Обработка задания приостановлена пользователем	---	01:24:19	05.07.2025 18:41:37	Joint search of DLS spectra with Gerasim project 3.17	wu_e1072_n14_distr_rnd_diag_to_sp_dt_wu1697662_0

RakeSearch volunteer distributed computing project (BOINC platform)

- real performance – **12 TFLOP/s** (by BOINC benchmark information), **6,8 TFLOP/s** (by http://boincstats.com information)

- users number – more than **2400** (269 now is active)
- hosts number – more than **12500** (709 now is active)
- team number – **194** (63 now is active)

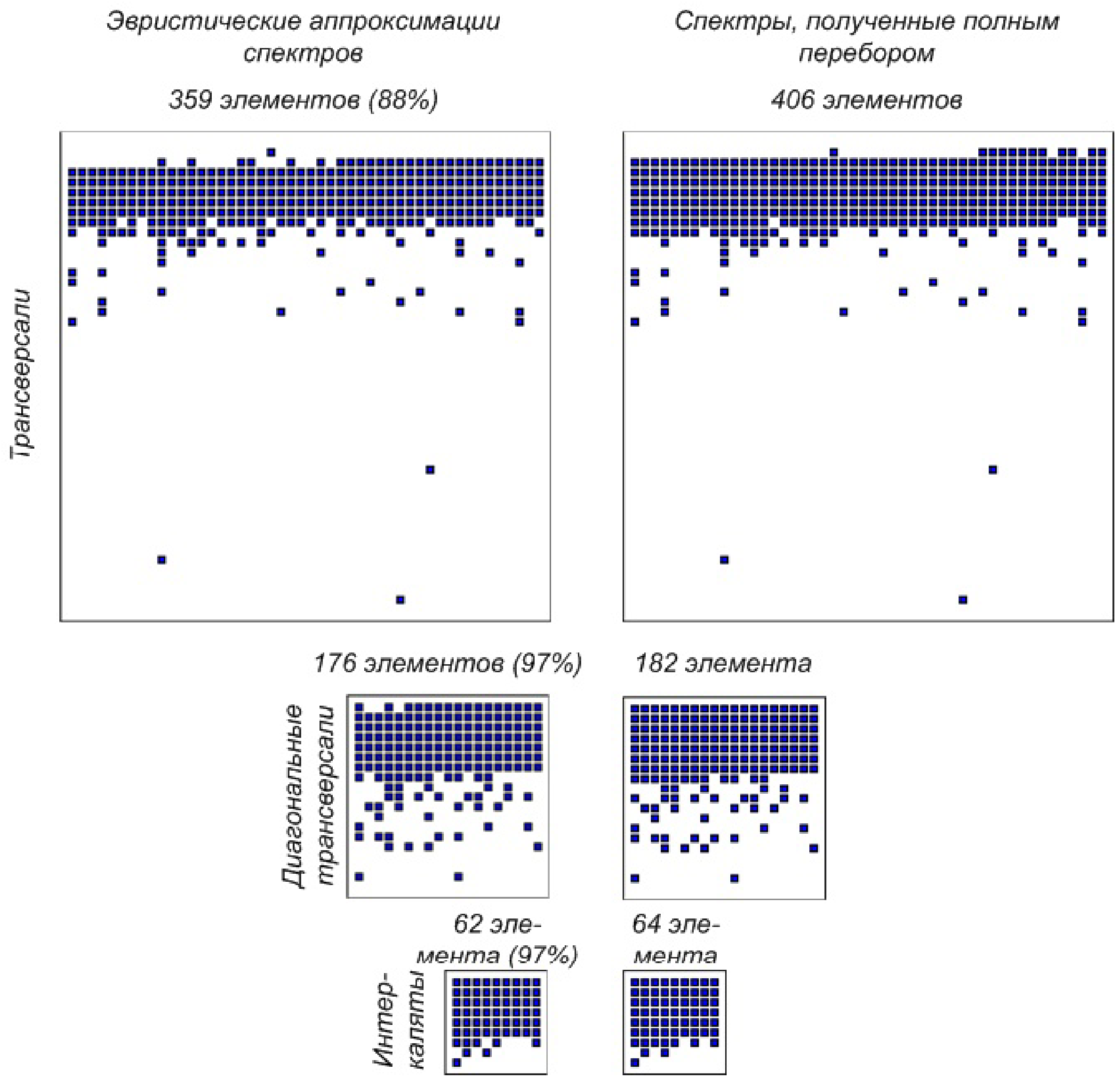
- https://rake.boincfast.ru/rakesearch/

Spectra of small order: Brute Force approach

DLS order	1	4	5	6	7	8	9
Transversals	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*
Diagonal transversals	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*
Intercalates	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*
ODLS	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*	*BF*

- computing experiment for order N=9 required 5 months of calculations using volunteer computing projects (Gerasim@Home + RakeSearch) with real performance 7—8 TFLOP/s and a set of parallel worked subprojects

Evaluation of the effectiveness of heuristic approximation methods using the example of spectra of order N=9



General strategy for heuristic spectra construction

- getting base spectrum;
- expanding of the base spectrum.

Base spectra:

- from different spectra for DLS (for example, construction of the transversals spectrum after diagonal transversals spectrum);
- from DLS of some special type (Brown, Gergely, (semi)cyclic, …).

Expanding of the base spectrum methods:

- neighborhoods after 1 intercalate/loop rotation;
- diagonalizing.

Spectrum expanding methods

Neighborhoods analyzing (abbr. **M1**)

- spectrum closure, parallelized by different squares in spectrum

Square by square neighborhoods analyzing (abbr. **M2**)

- separate closure of spectra starting from the squares of the original spectrum, only for small orders (N<13)

Diagonalizing (abbr. **d**)

- full (abbr. **fd**, for all squares in spectrum), only for small orders (N<14)
- partial (heuristic, abbr. **hd**) – some first experiments, for order N=14 (now) and higher (in future)

Methods based on movement in space of squares (in future for orders N>15-16, constructing full spectra is difficult)

- greedy movement «down» and «up»
- random walks
- weighted random walks
- other discrete combinatorial optimization methods

Constructing DLS spectra: M1 method

NSCF 2022

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16
Transversals	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	To Do	To Do	x
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	M1	x
Intercalates	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	To Do	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x		

NSCF 2023

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16
Transversals	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1 *	To Do	x
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	M1	x
Intercalates	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x		

NSCF 2024

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
Transversals	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	To Do	x										
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	M1	x										
Intercalates	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x												

Constructing DLS and LS spectra: M1 method

DLS spectra

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
Transversals	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	To Do	x										
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	M1	x										
Intercalates	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x												

LS spectra (NSCF 2024)

LS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	
Transversals	BF	BF	BF	BF	BF	BF	BF	To Do	...											
Intercalates	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	To Do	...

LS spectra (GRID 2025)

LS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	
Transversals	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	To Do	...											
Intercalates	BF	BF	BF	BF	BF	BF	BF	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	M1	To Do	...

Constructing DLS spectra: M2 method

NSCF 2022

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16
Transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M1	To Do	To Do	x
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	M1	M1	x
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	To Do	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x		

NSCF 2023

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16
Transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	x			
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	x		
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	M2	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x		

NSCF 2024

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
Transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	x													
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	x												
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2 *	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x												

Constructing DLS and LS spectra: M2 method

DLS spectra

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
Transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	x													
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	x												
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2 *	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x												

LS spectra (NSCF 2024)

LS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	
Transversals	BF	BF	BF	BF	BF	BF	BF	To Do												
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	To Do	...

LS spectra (GRID 2025)

LS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
Transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2 *	To Do													
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2 *	To Do	...

More detailed representation of the experiments

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16
Transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M1	To Do	To Do	x
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	M1	M1	x
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	To Do	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s	To Do	x?	x?	

Constructing DLS spectra: diagonalizing

NSCF 2022

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16
Transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M1	To Do	To Do	x
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	fd	fd	fd	To Do	To Do	To Do	x
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	To Do	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x		

NSCF 2023

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16
Transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M1	M1 *	To Do	x
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	fd	fd	fd	fd	x?	x?	x
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	M2	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x		

NSCF 2024, GRID 2025

DLS order	1	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	
Transversals	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M1	M1	To Do	x										
Diagonal transversals	BF	BF	BF	BF	BF	BF	BF	fd	fd	fd	fd	hd*	To Do	...										
Intercalates	BF	BF	BF	BF	BF	BF	BF	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2	M2 *	To Do	...
ODLS	BF	BF	BF	BF	BF	BF	BF	s	s	s*	To Do	x												

New results (NSCF’21 → NSCF’22 → NSCF’23 → NSCF’24 → GRID’25)

Cardinalities of the heuristic approximations of the **DLS** spectra:

- transversals in **DLS** (https://oeis.org/A344105):
  - a(9)≥359 → a(9)=406 (2022)
  - ? → a(10)≥442 (2022)
  - ? → a(11)≥1158 → a(11)≥5081 (2022)
  - a(12)≥22407 → a(12)≥23113 (2022)
  - ? → a(13)≥74315 → a(13)≥75891 (2023)
  - ? → ? → a(14)≥281825 → a(14)≥290681 → **in process…** (2024)

- diagonal transversals in **DLS** (https://oeis.org/A345370):
  - a(9)≥176 → a(9)=182 (2022)
  - ? → a(10)≥736 (2022)
  - a(11)≥353 → a(11)≥1242 (2022)
  - a(12)≥17641 → a(12)≥17693 (2022)
  - ? → a(13)≥12050 → a(13)≥18241 (2023)
  - ? → a(14)≥281067 → a(14)≥294053 → a(14)≥297090 → **a(14)≥377674** → **in process…** **(2025)**
  - ? → a(15)≥1958394 → **ToDo…** (2022)

New results (NSCF'21 → NSCF'22 → NSCF'23 → NSCF'24 → GRID'25)

Cardinalities of the heuristic approximations of the **DLS** spectra:

- intercalates in **DLS** (https://oeis.org/A345760):
  - a(9)≥62 → a(9)=64 (2022)
  - ? → a(10)≥88 → a(10)≥93 → a(10)≥98 (2024)
  - ? → a(11)≥100 → a(11)≥101 → a(11)≥145 (2024)
  - ? → a(12)≥210 → a(12)≥210 → a(12)≥259 (2024)
  - ? → a(13)≥152 → a(13)≥152 → a(13)≥197 → **a(13)≥200** **(2025)**
  - ? → ? → a(14)≥337 → a(14)≥362 (2024)
  - ? → ? → a(15)≥18 → a(15)≥536 (2024)
  - ? → ? → a(16)≥11 → a(16)≥789 → **a(16)≥792** **(2025)**
  - ? → ? → a(17)≥7 → a(17)≥685 (2024)
  - ? → ? → ? → a(18)≥419 → **a(18)≥535** **(2025)**
  - ? → ? → a(19)≥15 → a(19)≥447 (2024)
  - ? → ? → ? → a(20)≥1009 → **a(20)≥1011** **(2025)**
  - ? → ? → ? → a(21)≥740 → **a(21)≥772** **(2025)**
  - ? → ? → ? → a(22)≥737 → **a(22)≥872** **(2025)**
  - ? → ? → ? → a(23)≥885 → **a(23)≥887** **(2025)**
  - ? → ? → ? → a(24)≥1610 → **a(24)≥1845** → **ToDo…** **(2025)**
  - ? → ? → ? → ? → **a(25)≥1703** → **ToDo…** **(2025)**
  - ? → ? → ? → ? → **a(26)≥1266** → **ToDo…** **(2025)**
  - ? → ? → ? → ? → **a(27)≥1337** → **ToDo…** **(2025)**
  - ? → ? → ? → ? → **a(28)≥2829** → **ToDo…** **(2025)**

New results (NSCF’21 → NSCF’22 → NSCF’23 → NSCF’24 → GRID’25)

Cardinalities of the heuristic approximations of the DLS spectra:

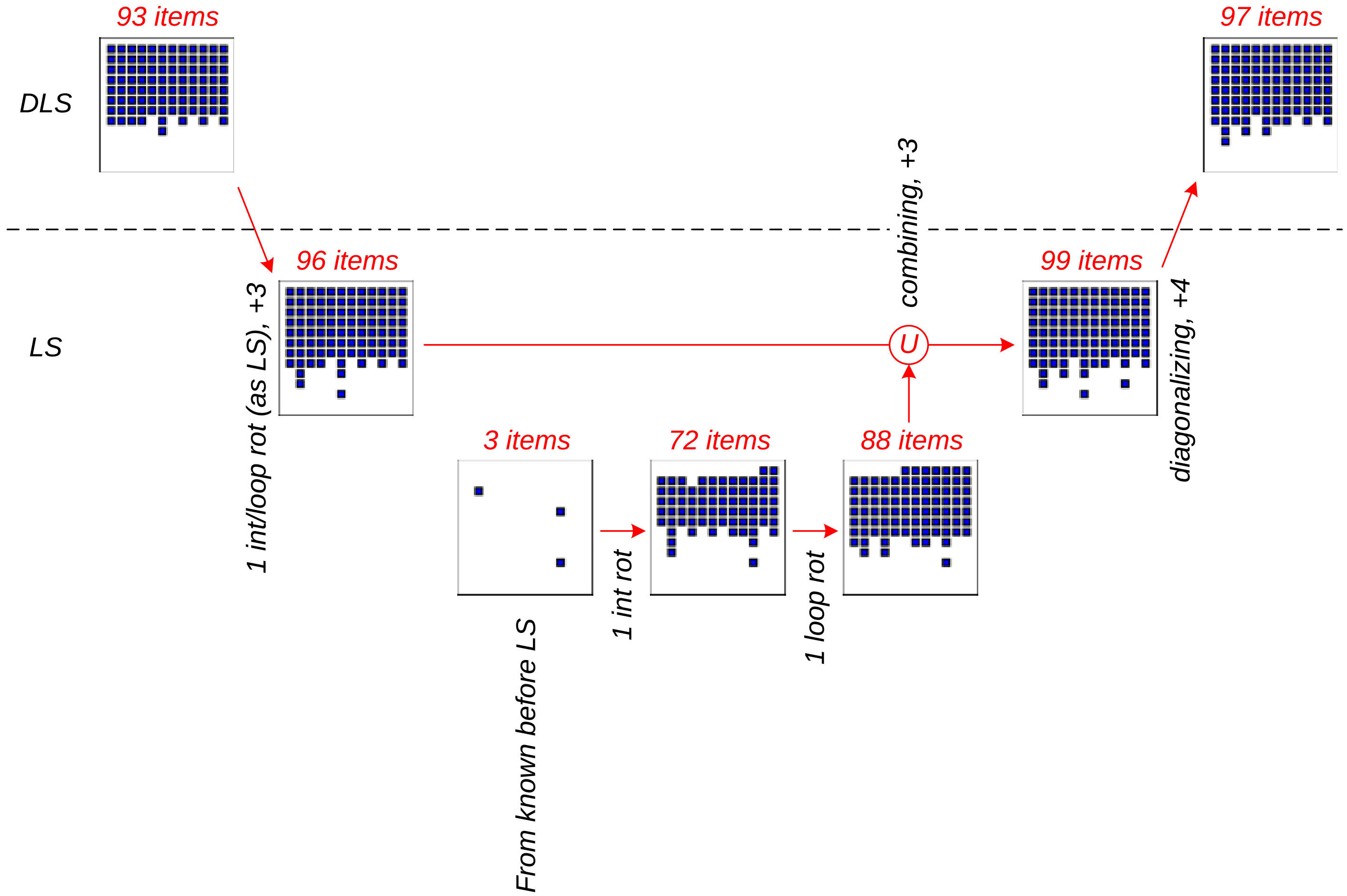
- number of ODLS in DLS (https://oeis.org/A345761):
  - a(10)≥10 (2021)
  - a(11)≥36 → a(11)≥39 → a(11)≥112 (2023)
  - a(12)≥2782 → a(12)≥4930 → a(12)≥5522 → a(12)≥5842 → → **a(12)≥5995** → **in process…** **(2025)**

New sup/inf values corresponds to the archived cardinalities

From DLS to LS and back to DLS

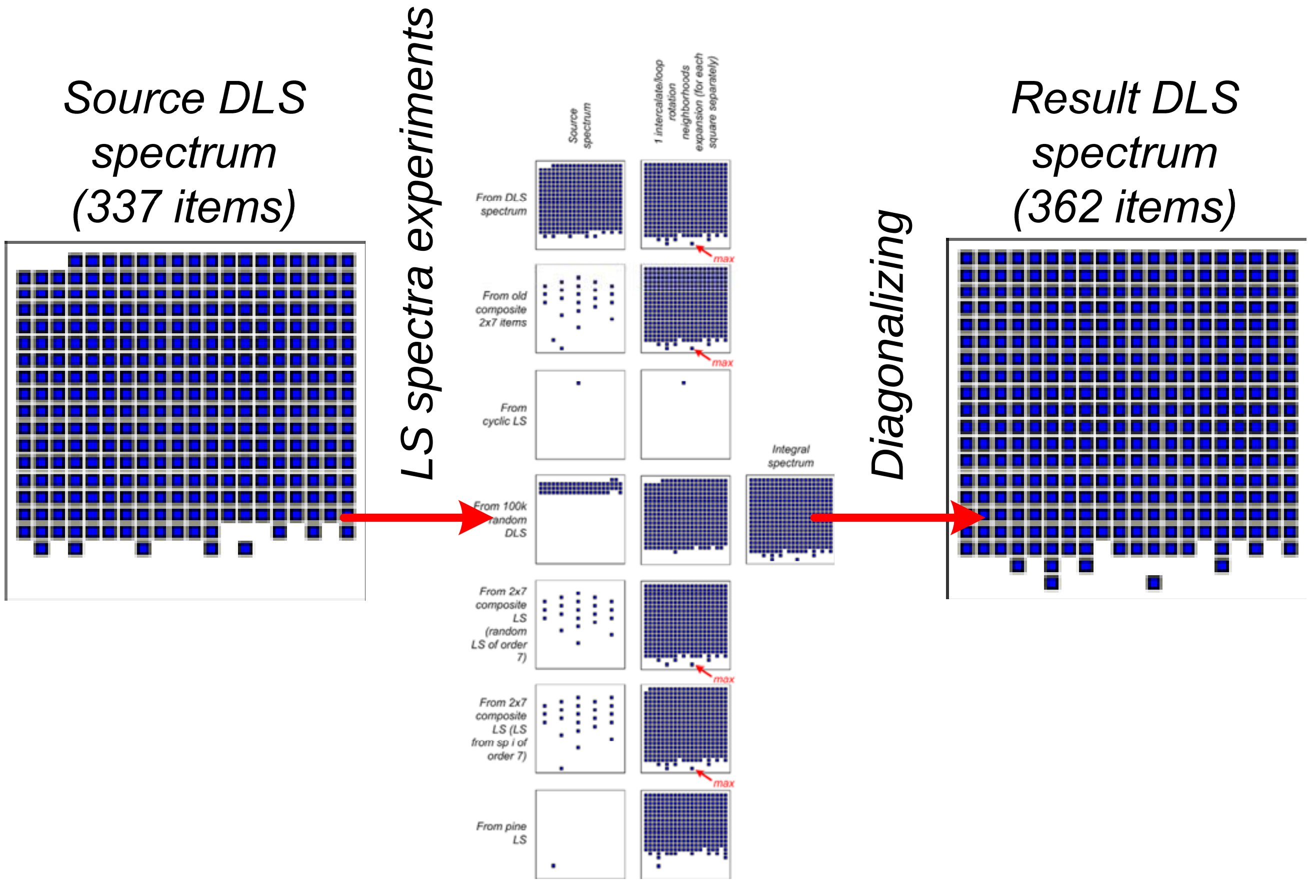
- base DLS spectrum
- expanding DLS spectrum (working with DLS as LS) – resulting **LS** spectrum
- diagonalizing LS from LS spectrum – resulting **DLS** spectrum

Order N=10 example



From LS to DLS spectrum using diagonalization example for order N=14

- base DLS spectrum
- expanding DLS spectrum (working with DLS as LS) – resulting **LS** spectrum
- diagonalizing LS from LS spectrum – resulting **DLS** spectrum



- difficult (impossible?) from orders N≥16
- randomization of diagonalizing procedure?

New results (GRID'25)

Cardinalities of the heuristic approximations of the **LS** spectra:

- transversals in **LS** (https://oeis.org/A309344):
  - **a(9)≥407** **(2024)**
  - **a(10)≥463** **(2025)**
  - **a(11)≥6437** **(2025)**
  - **a(12)≥23664** → **in process…** **(2025)**

from DLS spectra (weak estimates):

- a(13)≥75891 → **ToDo…** (2023)
- a(14)≥290681 → **ToDo…** (2024)

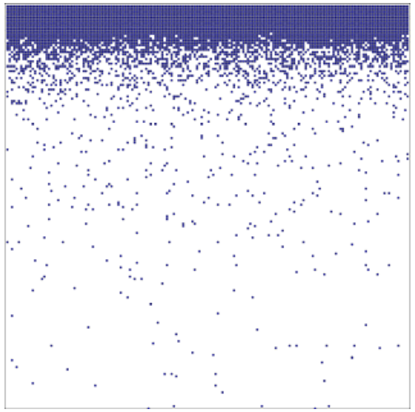
New results (NSCF’24 → GRID’25)

Cardinalities of the heuristic approximations of the **LS** spectra:

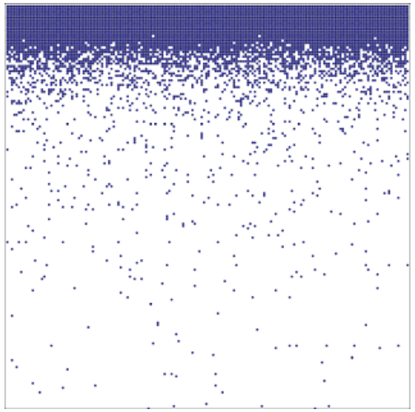
- intercalates in **LS** (https://oeis.org/A368182):
  - a(9)≥64 (2024)
  - a(10)≥103 (2024)
  - a(11)≥145 (2024)
  - a(12)≥259 (2024)
  - a(13)≥197 → **a(13)≥200** **(2025)**
  - a(14)≥362 (2024)
  - a(15)≥536 (2024)
  - a(16)≥794 (2024)
  - a(17)≥705 (2024)
  - a(18)≥655 (2024)
  - a(19)≥469 (2024)
  - a(20)≥1362 (2024)
  - a(21)≥740 → **a(21)≥985** **(2025)**
  - a(22)≥737 → **a(22)≥1450** **(2025)**
  - a(23)≥885 → **a(23)≥981** **(2025)**
  - a(24)≥1610 → **a(24)≥2560** → **ToDo…** **(2025)**
  - **a(25)≥1703** → **ToDo…** **(2025)**
  - **a(26)≥1266** → **ToDo…** **(2025)**

  from DLS spectra (weak estimates):
  - **a(27)≥1337** → **ToDo…** **(2025)**
  - **a(28)≥2829** → **ToDo…** **(2025)**

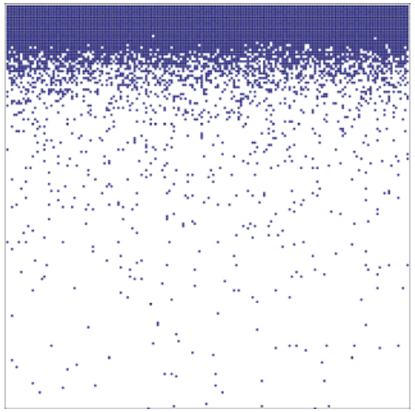
Approximation of the spectrum of ODLS number for DLS of order 12



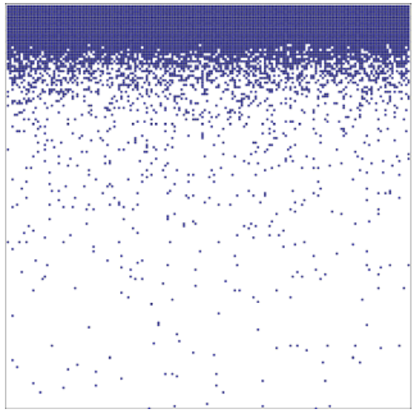
Lower part of the spectrum (NSCF’22, 4897 items)



Lower part of the spectrum (NSCF’23, 5522 items)



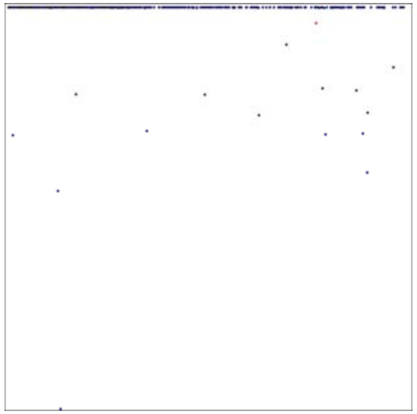
Lower part of the spectrum (NSCF’24, 5844 items)



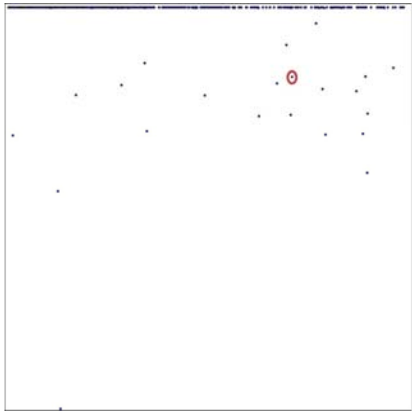
Lower part of the spectrum (GRID’25, **5995 items**)

- lower part of the spectrum — 1 threaded program implementation on the Core i7 4770;
- higher part of the spectrum — square by square in the RakeSearch (https://rake.boincfast.ru/rakesearch/) volunteer computing project for DLS with big number of diagonal transversals;
- http://evatutin.narod.ru/spectra/spectrum_dls_odls_n12_xxxx_known_items.txt

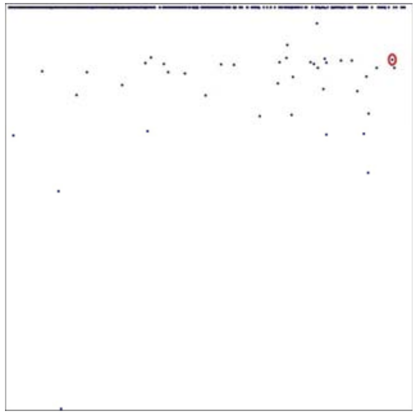
Approximation of the spectrum of ODLS number for DLS of order 12



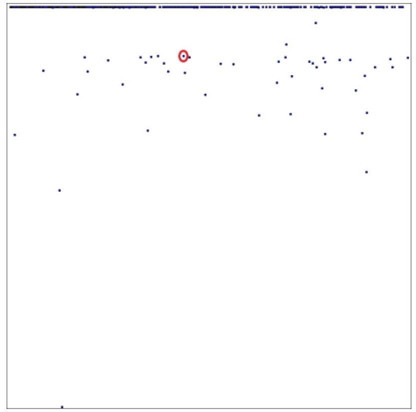
Higher part of the spectrum (NSCF’22, 4897 items)



Higher part of the spectrum (NSCF’23, 5522 items)



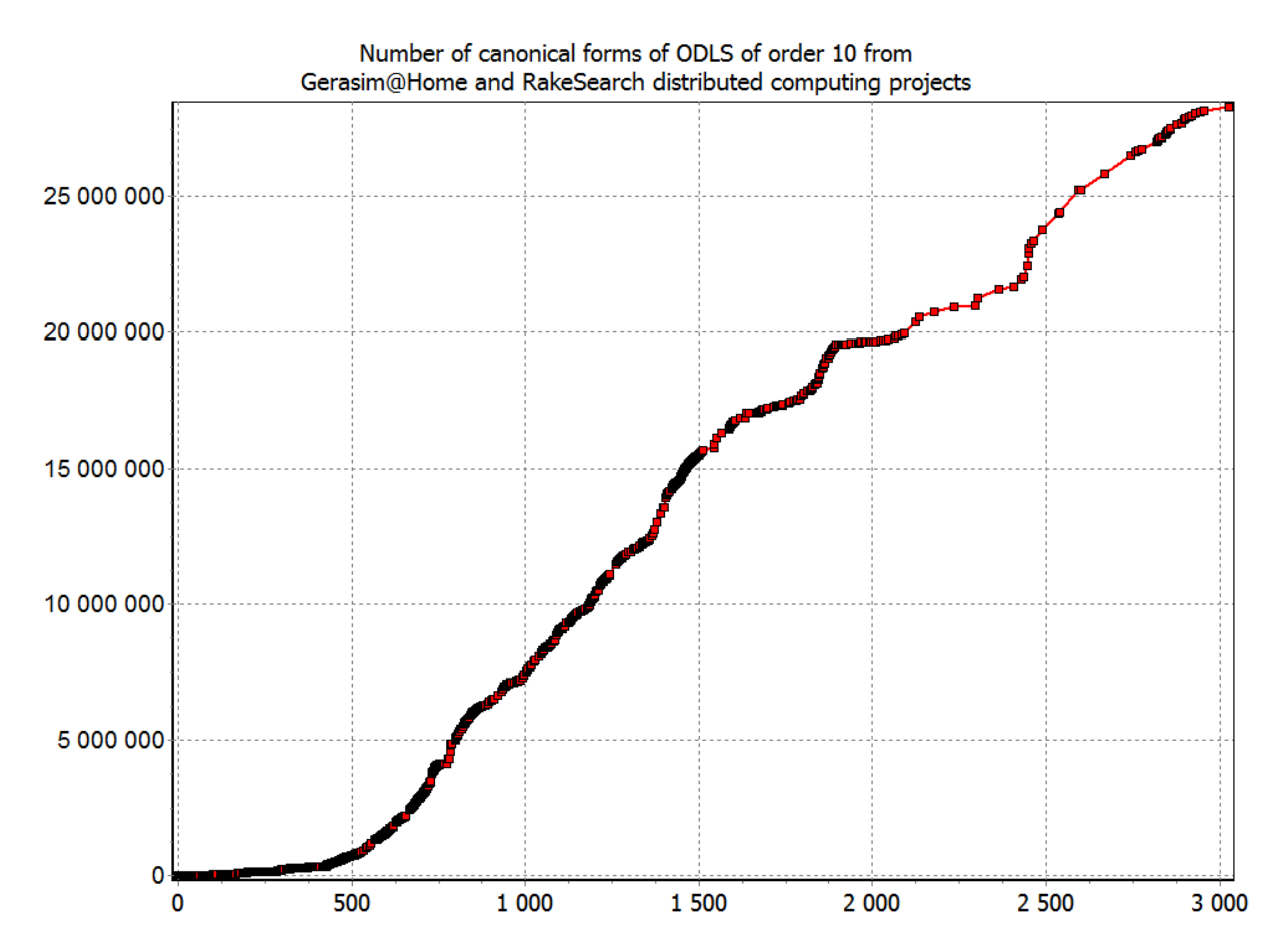
Higher part of the spectrum (NSCF’24, 5844 items)



Higher part of the spectrum (GRID’25, **5995 items**)

- lower part of the spectrum — 1 threaded program implementation on the Core i7 4770;
- higher part of the spectrum — square by square in the RakeSearch (https://rake.boincfast.ru/rakesearch/) volunteer computing project for DLS with big number of diagonal transversals;
- http://evatutin.narod.ru/spectra/spectrum_dls_odls_n12_xxxx_known_items.txt

Searching for ODLS of order 10 experiment results



For future:

- combination of methods for searching for ODLS and heuristic approximation of spectra;
- extended parastrophic transformations;
- …

Some new theoretical results

- maximum number a(N) of intercalates in LS of order $N = 2^n$
  - a(N) = N^2*(N-1)/4 (R. Bean, 2004)
  - How? For what type of squares?
  - Now I know: composite squares method 2x2 → 4x4 → 8x8 → … (**formula confirmed empirically!**)
  - Also: a(n) = n^2*(n-1)/4 **= A016152(log2(n)), new interconnection between different numerical series!**

0	1
1	0

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
1	0	3	2	5	4	7	6	9	8	B	A	D	C	F	E
2	3	0	1	6	7	4	5	A	B	8	9	E	F	C	D
3	2	1	0	7	6	5	4	B	A	9	8	F	E	D	C
4	5	6	7	0	1	2	3	C	D	E	F	8	9	A	B
5	4	7	6	1	0	3	2	D	C	F	E	9	8	B	A
6	7	4	5	2	3	0	1	E	F	C	D	A	B	8	9
7	6	5	4	3	2	1	0	F	E	D	C	B	A	9	8
8	9	A	B	C	D	E	F	0	1	2	3	4	5	6	7
9	8	B	A	D	C	F	E	1	0	3	2	5	4	7	6
A	B	8	9	E	F	C	D	2	3	0	1	6	7	4	5
B	A	9	8	F	E	D	C	3	2	1	0	7	6	5	4
C	D	E	F	8	9	A	B	4	5	6	7	0	1	2	3
D	C	F	E	9	8	B	A	5	4	7	6	1	0	3	2
E	F	C	D	A	B	8	9	6	7	4	5	2	3	0	1
F	E	D	C	B	A	9	8	7	6	5	4	3	2	1	0

Some new theoretical results

- maximum number a(N) of intercalates in LS of order $N = 2^n - 1$
  - $a(N) = N*(N-1)*(N-3)/4$ (R. Bean, 2004)
  - How? For what type of squares? Unclear… ☹
  - Results of heuristic search: **formula confirmed empirically for orders N in {3, 7, 15}!**
  - Confirmation for N=31 in future?
  - New formula: a(n) = (2^n – 1)*(2^n – 2)*(2^n – 4)/4 →

    New numerical series 0, 0, 42, 630, 6510, 58590, 496062, 4080510, 33097470, 266604030, 2140150782, 17150523390, 137321541630, 1099041923070, 8794214088702, 70361228214270, 562919889108990, 4503479369203710, 36028315984461822, 288228452010033150, 2305835312639639550, … →

    New formula A092237(2^n-1) = 42*A006096(n)
  - **New interconnection between Gaussian binomial coefficients [n,3] and intercalates in LS!**
  - Also: a(N) = N*(N-1)*(N-3)/4 **= A006096(log2(N+1))*42, new interconnection between different numerical series!**

Some new theoretical results

- number of intercalates in **pine LS**
  - New type of squares (R. Bean (2004) used it, but without careful definition) – pine LS (none canonical composite squares of order N=2k from cyclic subsquares of order k).
  - Cyclic subsquares of order k hasn't intercalates, but suddenly…
  - Record number of intercalates in LS for some orders: N=10 (R. Bean, 2004), **N in {18, 24, 26, 28} (E.I. Vatutin, 2024)**
  - **Same number of intercalates for selected order N.**
  - **Hypothesis 1: if N=4k+2, number of intercalates a(n) = a(N/2)= a(2k+1) = (N/2)^3 = (2k+1)^3 = A016755((n-1)/2) (empirically verified for N<29).**
  - **Hypothesis 2: if N=4k number of intercalates a(n) = a(N/2) = a(2k) = (N/2)^2 + (N/2)^3 = 4*k^2 + 8*k^3 = (2k)^2 * (2k+1) = 2*A089207(n/2) = 4*A099721(n/2) (verified for N<29).**
  - **New interconnection between different numerical series!**

Some new theoretical results

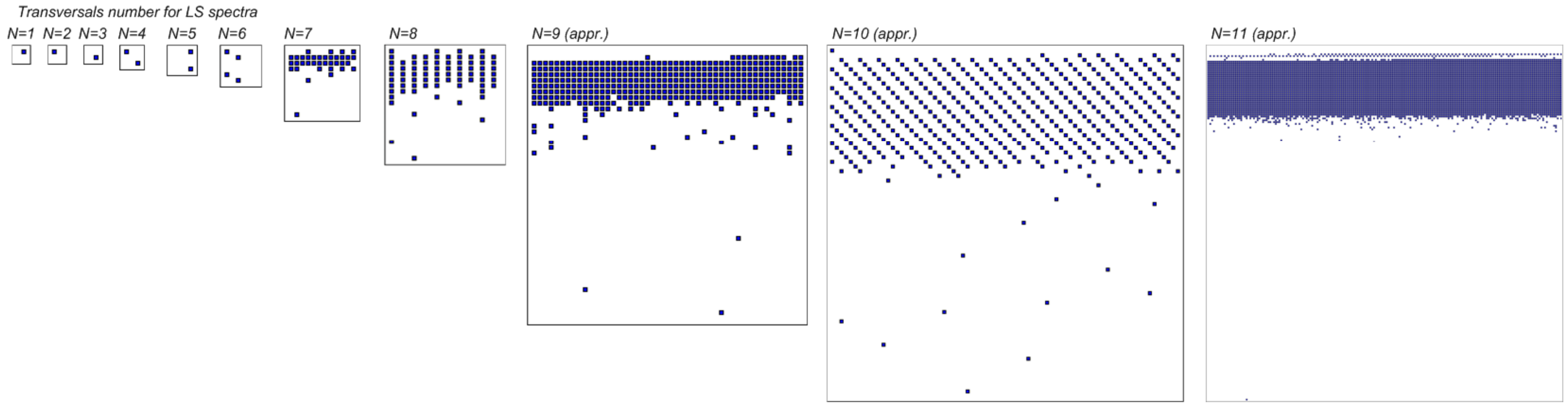
- upper and lower bounds for maximal number of intercalates a(n) in LS
  - **a(n) <= (n*(n-1))^2 in theory, a(n) << (n*(n-1))^2 in practice, a(n) < O(n^4) asymptotically! Polynomial, not exponential grows!**
  - **o(k1*n^3) < a(n), k1=1/8, for orders N=2*k due to pine LS existence!**

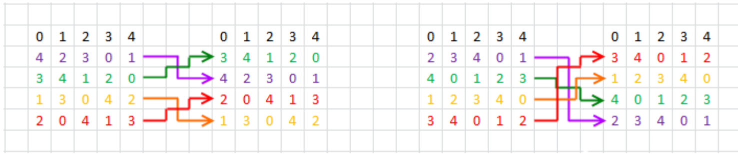
0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
1	2	3	4	5	6	7	0	F	8	9	A	B	C	D	E
2	3	4	5	6	7	0	1	E	F	8	9	A	B	C	D
3	4	5	6	7	0	1	2	D	E	F	8	9	A	B	C
4	5	6	7	0	1	2	3	C	D	E	F	8	9	A	B
5	6	7	0	1	2	3	4	B	C	D	E	F	8	9	A
6	7	0	1	2	3	4	5	A	B	C	D	E	F	8	9
7	0	1	2	3	4	5	6	9	A	B	C	D	E	F	8
8	9	A	B	C	D	E	F	0	1	2	3	4	5	6	7
9	A	B	C	D	E	F	8	7	0	1	2	3	4	5	6
A	B	C	D	E	F	8	9	6	7	0	1	2	3	4	5
B	C	D	E	F	8	9	A	5	6	7	0	1	2	3	4
C	D	E	F	8	9	A	B	4	5	6	7	0	1	2	3
D	E	F	8	9	A	B	C	3	4	5	6	7	0	1	2
E	F	8	9	A	B	C	D	2	3	4	5	6	7	0	1
F	8	9	A	B	C	D	E	1	2	3	4	5	6	7	0

Some new theoretical results

- hypothesizes about the number of transversals in LS same as in DLS
    - **T(N) = 4m for all N=4k+2**
    - **T(N) = 2m for all N=2k**

  (both found empirically, see picture above)





Thank you for your attention!

Thanks to all the volunteers who take part in the RakeSearch volunteer distributed computing project!

WWW: http://evatutin.narod.ru, https://rake.boincfast.ru/rakesearch/
E-mail: evatutin@rambler.ru