Southwest State University
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ENUMERATING CYCLIC AND PANDIAGONAL LATIN SQUARES AND THEIR PROPERTIES

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Pereslavl-Zaleskiy, 2020

What are Latin and diagonal Latin squares?

$A = \left\| a_{ij} \right\|$

$i, j = \overline{1, N}$

$N = |S|$

$S = \{0, 1, 2, \ldots, N-1\}$

$\forall i, j, k = \overline{1, N}, j \neq k : (a_{ij} \neq a_{ik}) \wedge (a_{ji} \neq a_{ki})$

$\forall i, j = \overline{1, N}, i \neq j : (a_{ii} \neq a_{jj}) \wedge (a_{N-i+1, N-i+1} \neq a_{N-j+1, N-j+1})$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 9 & 4 & 3 & 6 & 7 & 5 & 0 & 8 \\
2 & 9 & 3 & 1 & 7 & 0 & 5 & 8 & 4 & 6 \\
3 & 4 & 1 & 2 & 8 & 7 & 9 & 6 & 5 & 0 \\
4 & 3 & 5 & 9 & 2 & 1 & 8 & 0 & 6 & 7 \\
5 & 6 & 4 & 8 & 1 & 2 & 0 & 9 & 7 & 3 \\
6 & 5 & 8 & 7 & 0 & 3 & 2 & 1 & 9 & 4 \\
7 & 8 & 6 & 0 & 9 & 4 & 1 & 2 & 3 & 5 \\
8 & 7 & 0 & 5 & 6 & 9 & 3 & 4 & 1 & 2 \\
9 & 0 & 7 & 6 & 5 & 8 & 4 & 3 & 2 & 1
\end{pmatrix}$$

Normalized LS of order 10

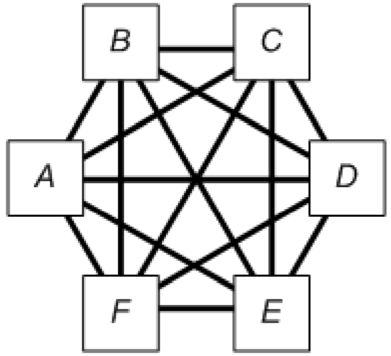
$$N! \times (N-1)!$$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
7 & 2 & 4 & 9 & 0 & 6 & 5 & 1 & 3 & 8 \\
8 & 3 & 6 & 7 & 5 & 9 & 0 & 2 & 4 & 1 \\
2 & 6 & 8 & 5 & 1 & 7 & 4 & 0 & 9 & 3 \\
5 & 8 & 9 & 1 & 7 & 0 & 3 & 4 & 6 & 2 \\
9 & 4 & 1 & 2 & 8 & 3 & 7 & 6 & 0 & 5 \\
4 & 7 & 5 & 6 & 9 & 1 & 8 & 3 & 2 & 0 \\
3 & 0 & 7 & 8 & 2 & 4 & 1 & 9 & 5 & 6 \\
6 & 5 & 0 & 4 & 3 & 2 & 9 & 8 & 1 & 7 \\
1 & 9 & 3 & 0 & 6 & 8 & 2 & 5 & 7 & 4
\end{pmatrix}$$

Normalized DLS of order 10

$$(N-1)!$$

Why is it interesting?



Applied problems:

- experiment planning
- cryptography
- error correcting codes
- scheduling
- algebra, combinatorics, statistics, …

Mathematical problems:

- **existence of a triple of MOLS/MODLS of order 10 (or larger clique)**
- increasing world record of orthogonality characteristic for pseudo triple of MOLS (291/300) or MODLS (274/300)
- generating functions
- asymptotic behavior of combinatorial characteristics based on DLSs (OEIS)
- number theory (relations between different fields of knowledge)
- magic squares
- Sudoku (LS of order 9 with additional constraints)

Special types of LS/DLS

row[i] = cyclic_shift(row[i-1], d)

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

Cyclic LS (d=1)

0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4

Cyclic DLS (d=2)

- there are known different special types of LS/DLS (plane symmetric, central symmetric, …);
- all cyclic squares of small orders are pandiagonal.

Integer sequences (OEIS): N queens problem example

The OEIS Foundation is supported by donations from users of the OEIS and by a grant from the Simons Foundation.

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

n queens | Search | Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **n queens**

Displaying 1-10 of 708 results found. page 1 2 3 4 5 6 7 8 9 10 ... 71

Sort: relevance | references | number | modified | created Format: long | short | data

A000170 Number of ways of placing **n** nonattacking **queens** on an **n** X **n** board. +40 75

(Formerly M1958 N0775)

1, 1, 0, 0, 2, 10, 4, 40, 92, 352, 724, 2680, 14200, 73712, 365596, 2279184, 14772512, 95815104, 666090624, 4968057848, 39029188884, 314666222712, 2691008701644, 24233937684440, 227514171973736, 2207893435808352, 22317699616364044, 234907967154122528 (list; graph; refs; listen; history; edit; text; internal format)

OFFSET 0,5

COMMENTS For **n** > 3, a(**n**) is the number of maximum independent vertex sets in the **n** X **n** **queen** graph. - Eric W. Weisstein, Jun 20 2017

Number of nodes on level **n** of the backtrack tree for the **n** **queens** problem (a(**n**) = A319284(**n**, **n**)). - Peter Luschny, Sep 18 2018

REFERENCES M. Gardner, The Unexpected Hanging, pp. 190-2, Simon & Shuster NY 1969

Jieh Hsiang, Yuh-Pyng Shieh and Yao-Chiang Chen, The cyclic complete mappings counting problems, in Problems and Problem Sets for ATP, volume 02-10 of DIKU technical reports, G. Sutcliffe, J. Pelletier and C. Suttner, eds., 2002.

D. E. Knuth, The Art of Computer Programming, Volume 4, Pre-fascicle 5B, Introduction to Backtracking, 7.2.2. Backtrack programming. 2018.

Massimo Nocentini, "An algebraic and combinatorial study of some infinite sequences of numbers supported by symbolic and logic computation", PhD Thesis, University of Florence, 2019. See Ex. 67.

W. W. Rouse Ball and H. S. M. Coxeter, Mathematical Recreations and Essays, 13th ed., New York, Dover, 1987, pp. 166-172 (The Eight **Queens** Problem).

M. A. Sainte-Laguë, Les Réseaux (ou Graphes), Mémorial des Sciences Mathématiques, Fasc. 18, Gauthier-Villars, Paris, 1926, p. 47.

N. J. A. Sloane, A Handbook of Integer Sequences, Academic Press, 1973 (includes this sequence).

N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

R. J. Walker, An enumerative technique for a class of combinatorial problems, pp. 91-94 of Proc. Sympos. Applied Math., vol. 10, Amer. Math. Soc., 1960.

M. B. Wells, Elements of Combinatorial Computing. Pergamon, Oxford, 1971, p. 238.

LINKS Table of n, a(n) for n=0..27.

Jordan Bell, Brett Stevens, A survey of known results and research areas for **n-queens**, Discrete Mathematics, Volume 309, Issue 1, Jan 06 2009, Pages 1-31.

D. Bill, Durango Bill's The **N-Queens** Problem

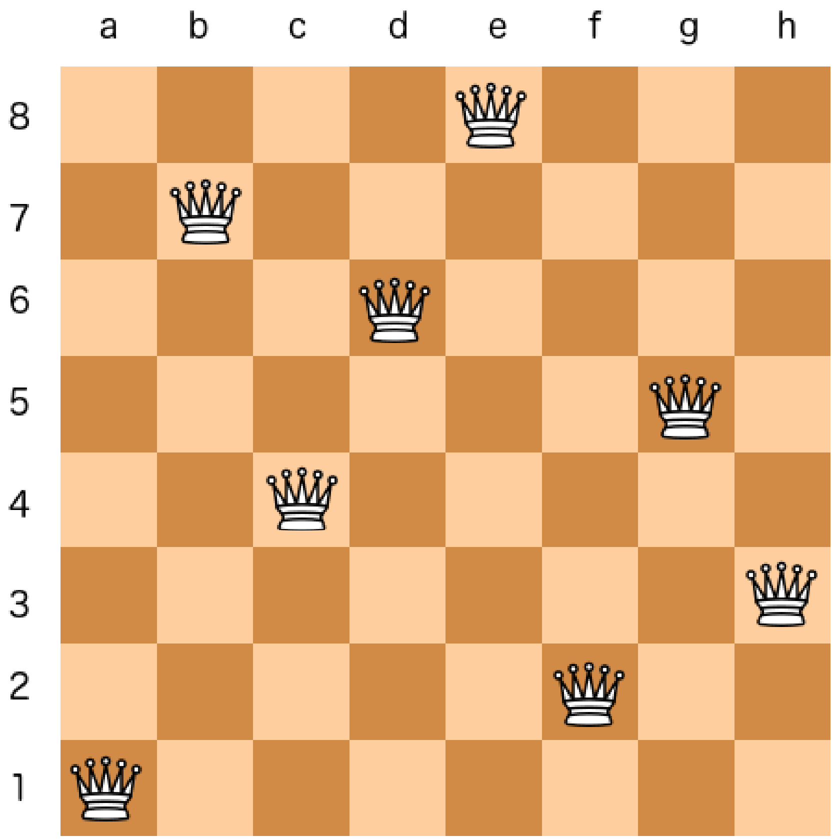
J. R. Bitner and E. M. Reingold, Backtrack programming techniques, Commun. ACM, 18 (1975), 651-656.

J. R. Bitner and E. M. Reingold, Backtrack programming techniques, Commun. ACM, 18 (1975), 651-656. [Annotated scanned copy]

P. Capstick and K. McCann, The problem of the **n queens**, apparently unpublished, no date (circa 1990?) [Scanned copy]

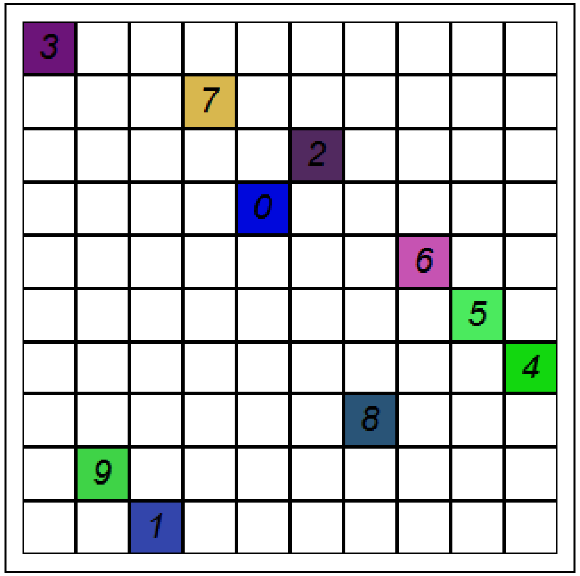
V. Chvatal, All solutions to the problem of eight **queens**

https://oeis.org/A000170

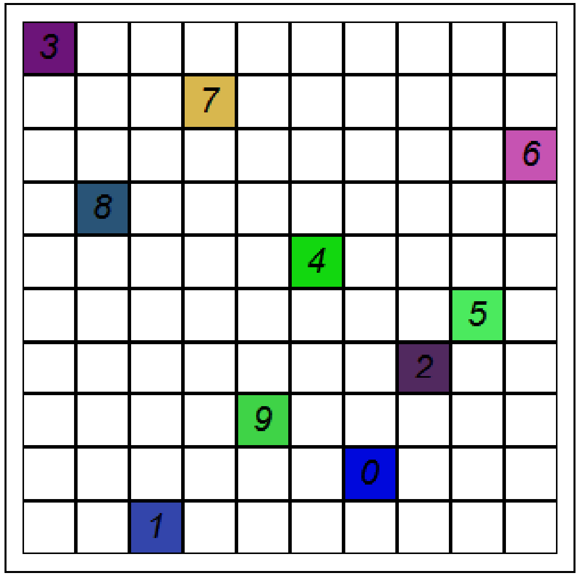


Integer sequences connected with transversals in DLS

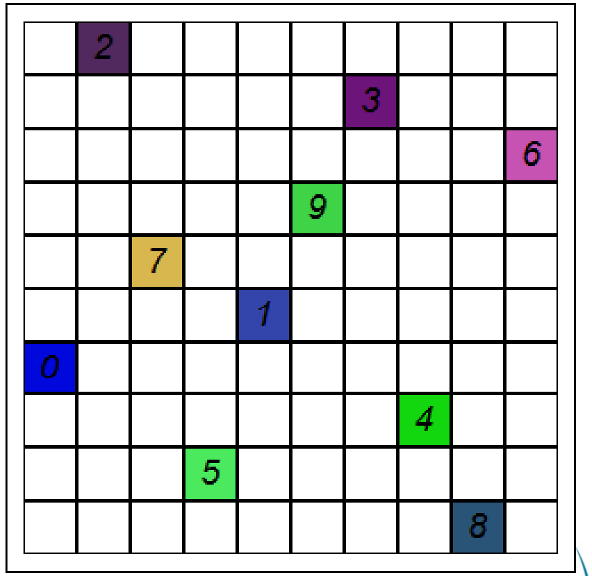
A287645 — Minimum number of transversals in a diagonal Latin square of order N (N<10)
A287644 — Maximum number of transversals in a diagonal Latin square of order N (N<10)
A287647 — Minimum number of diagonal transversals in a diagonal Latin square of order N (N<9)
A287648 — Maximum number of diagonal transversals in a diagonal Latin square of order N (N<9)



Transversal 1



Transversal 2



Transversal 3

Integer sequences connected with transversals in DLS: exactly known values

A287645 — **1, 0, 0, 8, 3, 32, 7, 8**, **68** (N<10)
A287644 — **1, 0, 0, 8, 15, 32, 133, 384**, **2241** (N<10)
A287647 — **1, 0, 0, 4, 1, 2, 0, 0**, **0** (N<10)
A287648 — **1, 0, 0, 4, 5, 6, 27, 120**, **333?** (N<10)

a(1)-a(8) — Brute Force for all DLS
a(9) — CFs generator for main classes of DLS based on X-fillings of diagonals and ESODLS schemas (currently ongoing, ends)

What about a(10), a(11), …? They are unknown

$$0 \le X_{\min}^{LS}(N) \le X_{\min}^{DLS}(N) \le X_{\max}^{DLS}(N) \le X_{\max}^{LS}(N)$$

Integer sequences connected with transversals in DLS: unknown values for high orders

A287645 — 1, 0, 0, 8, 3, 32, 7, 8, 68 (0 ≤ a(n) ≤ A287644(n) ≤ A090741(n))
A287644 — 1, 0, 0, 8, 15, 32, 133, 384, 2241, **≥5504** (a(n) <= A090741(n))
A287647 — 1, 0, 0, 4, 1, 2, 0, 0, 0 (a(n) <= A287648(n) <= A007016(n))
A287648 — 1, 0, 0, 4, 5, 6, 27, 120, **≥333, ≥866, ≥4828, ≥24901, ≥131106, ≥364596, ≥389318** (a(n) <= A007016(n))

Upper and lower bounds can be expanded and strengthened!

Transversals in random Latin squares

A287645 (minimum number of transversals)
a(11) ≤ 3266->3255->3185->3175
a(12) ≤ 15462
a(13) ≤ 78253
a(14) ≤ 422312
a(15) ≤ 2415635
a(16) ≤ 14689972
...

A287644 (maximum number of transversals)
a(11) ≥ 3867
a(12) ≥ 16600
a(13) ≥ 80999 (a(13) ≥ 1030367 for cyclic squares)
a(14) ≥ 428296
a(15) ≥ 2429398
a(16) ≥ 14720910
...

- 16 days on Core i7 4770 CPU, special program implementation of DLX (enumeration only without covers collecting)
- https://vk.com/wall162891802_1449

Diagonal transversals in random Latin squares

A287647 (minimum number of diagonal transversals)
a(11) ≤ 324
a(12) ≤ 1816
...

A287648 (maximum number of diagonal transversals)
a(11) ≥ 550
a(12) ≥ 2200
...

- 16 days on Core i7 4770 CPU, special program implementation of DLX (enumeration only without covers collecting)
- https://vk.com/wall162891802_1449

Cyclic Latin squares: one of special types of LS

0	1	2	3	4	5	6
0	1	2	3	4	5	6
0	1	2	3	4	5	6
0	1	2	3	4	5	6
0	1	2	3	4	5	6
0	1	2	3	4	5	6
0	1	2	3	4	5	6

d=0

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

d=1

0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4

d=2

0	1	2	3	4	5	6
3	4	5	6	0	1	2
6	0	1	2	3	4	5
2	3	4	5	6	0	1
5	6	0	1	2	3	4
1	2	3	4	5	6	0
4	5	6	0	1	2	3

d=3

0	1	2	3	4	5	6
4	5	6	0	1	2	3
1	2	3	4	5	6	0
5	6	0	1	2	3	4
2	3	4	5	6	0	1
6	0	1	2	3	4	5
3	4	5	6	0	1	2

d=4

0	1	2	3	4	5	6
5	6	0	1	2	3	4
3	4	5	6	0	1	2
1	2	3	4	5	6	0
6	0	1	2	3	4	5
4	5	6	0	1	2	3
2	3	4	5	6	0	1

d=5

0	1	2	3	4	5	6
6	0	1	2	3	4	5
5	6	0	1	2	3	4
4	5	6	0	1	2	3
3	4	5	6	0	1	2
2	3	4	5	6	0	1
1	2	3	4	5	6	0

d=6

- 4 cyclic DLS, 6 cyclic LS for N=7

Number of transversals in cyclic Latin squares

1, 0, 3, 0, 15, 0, 133, 0, 2025, 0, 37851, 0, 1030367, 0, 36362925

or without zeroes:

1, 3, 15, 133, 2025, 37851, 1030367, 36362925

A006717 — Number of ways of arranging 2n+1 nonattacking semi-queens on a (2n+1) X (2n+1) toroidal board

Also the number of transversals of a cyclic Latin square of order 2n+1 and the number of orthomorphisms of the cyclic group of order 2n+1 (Ian Wanless, 2001).

- Number of transversals are same for all cyclic LS of given order N

Number of transversals in cyclic Latin squares: consequence

A090741 (maximum number of transversals in LS):

a(11) ≥ 37851
a(13) ≥ 1030367
a(15) ≥ 36362925
a(17) ≥ 1606008513
a(19) ≥ 87656896891
a(21) ≥ 5778121715415
a(23) ≥ 452794797220965
a(25) ≥ 41609568918940625
...

Lower bounds can be added to sequence A090741 in OEIS...

- Number of transversals are same for all cyclic LS of given order N

Number of transversals in cyclic diagonal Latin squares: consequence

a(11) ≥ 37851
a(13) ≥ 1030367
cyclic DLS are not exists for N=15
a(17) ≥ 1606008513
a(19) ≥ 87656896891
cyclic DLS are not exists for N=21
a(23) ≥ 452794797220965
a(25) ≥ 41609568918940625

Lower bounds can be added to sequence A287644 in OEIS...

- exists not for all orders N

Number of diagonal transversals in cyclic diagonal Latin squares

Minimal number of diagonal transversals in cyclic DLS of order N=2n+1: **1, 0, 5, 27, 0, 4523, 128818, 0, 204330233, 11232045257** (not presented in OEIS)

Maximal number of diagonal transversals in cyclic DLS of order N=2n+1 : **1, 0, 5, 27, 0, 4665, 131106, 0, 204995269, 11254190082** (not presented in OEIS)

Corresponding upper and lower bounds can be added to A287647 (weak) and A287648...

Enumerating the cyclic (diagonal) Latin squares

<u>DLS:</u>

1, 0, 0, 0, 2, 0, 4, 0, 0, 0, 8, 0, 10, 0, 0, 0, 14, 0, 16, 0, 0, 0, 20, 0, 10, 0, 0, 0, 26, 0, 28, …

Squares of this special type exists not for all orders N.

The number of positive integers k which are ≤ n and where k, k-1 and k+1 are each coprime to n (well known numerical series that directly not connected with Latin squares!).

<u>LS:</u>

1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, 8, 8, 16, 6, 18, 8, …

Euler totient function phi(N)!!!
Not connected directly with Latin squares! Can be calculated through Latin squares…

Euler totient function calculating

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_m^{\alpha_m}$$

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_m}\right)$$

$$6 = 2^1 \cdot 3^1, \varphi(6) = 6\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 2$$

- calculating by definition (enumerating coprimes (k,n), k<n, Euclid algorithm)
- calculating using factoring and formula
- new method: calculating by enumerating cyclic Latin squares of order n

Euler totient function calculating by enumerating cyclic Latin squares

0	1	2	3	4	5
0	1	2	3	4	5
0	1	2	3	4	5
0	1	2	3	4	5
0	1	2	3	4	5
0	1	2	3	4	5

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

0	1	2	3	4	5
2	3	4	5	0	1
4	5	0	1	2	3
0	1	2	3	4	5
2	3	4	5	0	1
4	5	0	1	2	3

0	1	2	3	4	5
3	4	5	0	1	2
0	1	2	3	4	5
3	4	5	0	1	2
0	1	2	3	4	5
3	4	5	0	1	2

0	1	2	3	4	5
4	5	0	1	2	3
2	3	4	5	0	1
0	1	2	3	4	5
4	5	0	1	2	3
2	3	4	5	0	1

0	1	2	3	4	5
5	0	1	2	3	4
4	5	0	1	2	3
3	4	5	0	1	2
2	3	4	5	0	1
1	2	3	4	5	0

$$\varphi(6) = 2$$

- polynomial algorithm ($t \sim O(N^2)$, $m \sim O(N)$) without multiplications and divisions

Euler totient function OEIS description slightly expanded

```
          phi(p*n) = phi(n)*(floor(((n + p - 1) mod p)/(p - 1)) + p - 1), for primes p. -
            Gary Detlefs, Apr 21 2012
          For odd n, a(n) = 2*A135303((n-1)/2)*A003558((n-1)/2) or phi(n) = 2*c*k; the Coach
            theorem of Pedersen et al. Cf. A135303. - Gary W. Adamson, Aug 15 2012
          G.f.: Sum_{n>=1} mu(n)*x^n/(1 - x^n)^2, where mu(n) = A008683(n). - Mamuka
            Jibladze, Apr 05 2015
          a(n) = n - cototient(n) = n - A051953(n). - Omar E. Pol, May 14 2016
          a(n) = lim_{s->1} n*zeta(s)*(Sum_{d divides n} A008683(d)/(e^(1/d))^(s-1)), for n >
            1. - Mats Granvik, Jan 26 2017
          Conjecture: a(n) = Sum_{a=1..n} Sum_{b=1..n} Sum_{c=1..n} 1 for n > 1. The sum is
            over a,b,c such that n*c - a*b = 1. - Benedict W. J. Irwin, Apr 03 2017
          a(n) = Sum_{j=1..n} gcd(j, n) cos(2*Pi*j/n) = Sum_{j=1..n} gcd(j, n)
            exp(2*Pi*i*j/n) where i is the imaginary unit. Notice that the Ramanujan's sum
            c_n(k) := Sum_{j=1..n, gcd(j, n) = 1} exp(2*Pi*i*j*k/n) gives a(n) = Sum_{k|n}
            k*c_(n/k)(1) = Sum_{k|n} k*mu(n/k). - Michael Somos, May 13 2018
          G.f.: x*d/dx(x*d/dx(log(Product_{k>=1} (1 - x^k)^( mu(k)/k^2)))), where mu(n) =
            A008683(n). - Mamuka Jibladze, Sep 20 2018
          a(n) = Sum_{d|n} A007431(d). - Steven Foster Clark, May 29 2019
          G.f. A(x) satisfies: A(x) = x/(1 - x)^2 - Sum_{k>=2} A(x^k). - Ilya Gutkovskiy, Sep
            06 2019
          a(n) >= sqrt(n/2) (Nicolas). - Hugo Pfoertner, Jun 01 2020
          a(n) > n/(exp(gamma)*log(log(n)) + 5/(2*log(log(n)))), except for n=223092870
            (Rosser, Schoenfeld). - Hugo Pfoertner, Jun 02 2020
EXAMPLE   G.f. = x + x^2 + 2*x^3 + 2*x^4 + 4*x^5 + 2*x^6 + 6*x^7 + 4*x^8 + 6*x^9 + 4*x^10 +
            ...
          a(8) = 4 with {1, 3, 5, 7} units modulo 8. a(10) = 4 with {1, 3, 7, 9} units modulo
            10. - Michael Somos, Aug 27 2013
          From Eduard I. Vatutin, Nov 01 2020: (Start)
          The a(5)=4 cyclic Latin squares with the first row in ascending order are:
            0 1 2 3 4   0 1 2 3 4   0 1 2 3 4   0 1 2 3 4
            1 2 3 4 0   2 3 4 0 1   3 4 0 1 2   4 0 1 2 3
            2 3 4 0 1   4 0 1 2 3   1 2 3 4 0   3 4 0 1 2
            3 4 0 1 2   1 2 3 4 0   4 0 1 2 3   2 3 4 0 1
            4 0 1 2 3   3 4 0 1 2   2 3 4 0 1   1 2 3 4 0
          (End)
MAPLE     with(numtheory): A000010 := phi; [ seq(phi(n), n=1..100) ]; # version 1
          with(numtheory): phi := proc(n) local i, t1, t2; t1 := ifactors(n)[2]; t2 :=
            n*mul((1-1/t1[i][1]), i=1..nops(t1)); end; # version 2
MATHEMATICA Array[EulerPhi, 70]
PROG      (Axiom) [eulerPhi(n) for n in 1..100]
          (MAGMA) [ EulerPhi(n) : n in [1..100] ]; // Sergei Haller (sergei(AT)sergei-
            haller.de), Dec 21 2006
          (PARI) {a(n) = if( n==0, 0, eulerphi(n))}; /* Michael Somos, Feb 05 2011 */
          (Sage)
          # euler_phi is a standard function in Sage.
          def A000010(n): return euler_phi(n)
          def A000010_list(n): return [ euler_phi(i) for i in range(1, n+1)]
          # Jaap Spies, Jan 07 2007
          (PARI) { for (n=1, 100000, write("b000010.txt", n, " ", eulerphi(n))); } \\ Harry
            J. Smith, Apr 26 2009
          (Sage) [euler_phi(n) for n in range(1, 70)]  # Zerinvary Lajos, Jun 06 2009
          (Maxima) makelist(totient(n), n, 0, 1000); /* Emanuele Munarini, Mar 26 2011 */
          (Haskell) a n = length (filter (==1) (map (gcd n) [1..n])) -- Allan C. Wechsler,
            Dec 29 2014
          (Python)
          from sympy.ntheory import totient
          print([totient(i) for i in range(1, 70)])  # Indranil Ghosh, Mar 17 2017
CROSSREFS Cf. A008683, A003434 (steps to reach 1), A007755, A049108, A002202 (values).
          Cf. A005277 (nontotient numbers). For inverse see A002181, A006511, A058277.
          Jordan function J_k(n) is a generalization - see A059379 and A059380 (triangle of
            values of J_k(n)), this sequence (J_1), A007434 (J_2), A059376 (J_3), A059377
```

Euler totient function calculating by enumerating cyclic Latin squares: practice implementation

```
C:\Projects\Функция Эйлера\EulerTotientFunction.exe

1 - factorization
2 - Euclid
3 - cyclic Latin squares
4 - cyclic Latin squares with 0 find
5 - cyclic Latin squares with 0 find, half of d values

   N   v1     v2     v3     v4     v5    t1     t2      t3      t4      t5
---------------------------------------------------------------------------
   1:   1 -    1 -    1 -    1 -    1   221     94     227     175     287
   2:   1 -    1 -    1 -    1 -    1    54    166      79     154      66
   3:   2 -    2 -    2 -    2 -    2   166    605     284     163      75
   4:   2 -    2 -    2 -    2 -    2   217    314     568     257      76
   5:   4 -    4 -    4 -    4 -    4   211    613     858     475     311
   6:   2 -    2 -    2 -    2 -    2   166    456    1007     583     266
   7:   6 -    6 -    6 -    6 -    6   221    846    1236     749     299
   8:   4 -    4 -    4 -    4 -    4   239    843    1438     783     308
   9:   6 -    6 -    6 -    6 -    6   229    873    1991     810     553
  10:   4 -    4 -    4 -    4 -    4   200    822    2019     916     447
  11:  10 -   10 -   10 -   10 -   10   263   1257    3255    1472     774
  12:   4 -    4 -    4 -    4 -    4   211    955    2575     949     501
  13:  12 -   12 -   12 -   12 -   12   145   1441    4198    1638    1161
  14:   6 -    6 -    6 -    6 -    6   145   1324    3745    1641     907
  15:   8 -    8 -    8 -    8 -    8   202   1360    3974    1508    1085
  16:   8 -    8 -    8 -    8 -    8   284   1457    4896    2004    1173
  17:  16 -   16 -   16 -   16 -   16   212   1514    6624    2427    1753
  18:   6 -    6 -    6 -    6 -    6   378   1992    5500    1988    1257
  19:  18 -   18 -   18 -   18 -   18   221   2191    8758    3168    2224
  20:   8 -    8 -    8 -    8 -    8   305   1913    6519    2370    1481
  21:  12 -   12 -   12 -   12 -   12   339   2433    8151    3044    1777
  22:  10 -   10 -   10 -   10 -   10   260   2125    8904    3197    2076
  23:  22 -   22 -   22 -   22 -   22   281   2838   12497    4231    2705
  24:   8 -    8 -    8 -    8 -    8   227   2294    8471    2935    1892
  25:  20 -   20 -   20 -   20 -   20   305   2898   12745    4352    2744
  26:  12 -   12 -   12 -   12 -   12   184   2877    9632    4558    2557
  27:  18 -   18 -   18 -   18 -   18   336   2844   10330    5129    2641
  28:  12 -   12 -   12 -   12 -   12   284   3095   10043    4512    2412
  29:  28 -   28 -   28 -   28 -   28   242   3660   14609    7220    3935
  30:   8 -    8 -    8 -    8 -    8   580   3602   12497    4820    2880
  31:  30 -   30 -   30 -   30 -   30   260   3911   21007    7259    4642
  32:  16 -   16 -   16 -   16 -   16   353   3551   17396    6108    4104
  33:  20 -   20 -   20 -   20 -   20   227   3802   19527    6616    3959
  34:  16 -   16 -   16 -   16 -   16   199   3975   20179    7265    4364
  35:  24 -   24 -   24 -   24 -   24   263   3835   21207    7193    4382
  36:  12 -   12 -   12 -   12 -   12   408   3926   17393    5794    3666
  37:  36 -   36 -   36 -   36 -   36   411   4703   27321   10110    6274
  38:  18 -   18 -   18 -   18 -   18   314   4540   19013    8792    5237
  39:  24 -   24 -   24 -   24 -   24   320   4721   20893    8716    5098
  40:  16 -   16 -   16 -   16 -   16   241   4497   22573    7700    4929
  41:  40 -   40 -   40 -   40 -   40   124   5349   35158   12189    7942
  42:  12 -   12 -   12 -   12 -   12   348   4706   23081    7713    4724
  43:  42 -   42 -   42 -   42 -   42   173   5213   38388   13023    8015
  44:  20 -   20 -   20 -   20 -   20   375   5141   28702   70279    5939
  45:  24 -   24 -   24 -   24 -   24   387   5364   80872   10152    6573
  46:  22 -   22 -   22 -   22 -   22   326   5751   80826   11729    7308
  47:  46 -   46 -   46 -   46 -   46   263   6253   91978   15556    9559
  48:  16 -   16 -   16 -   16 -   16   405   5558   66552   10185    6253
  49:  42 -   42 -   42 -   42 -   42   438   6422   90340   15226    9278
  50:  20 -   20 -   20 -   20 -   20   565   6495   80645   12095    7477
```

- slower than factorization based calculating
- slower than long arithmetic implementation (extimation)

Brief conclusion

- new numerical series was calculated
- new upper and lower bounds for some series was established
- interconnections between Latin squares and different type combinatorial objects was established
- new method of Euler totient function calculating based on Latin squares was proposed

Related work

Collecting CFs and new combinatorial structures search:

- triple of MODLS (is it exist?)
- different structures?

GPU implementation of transversal, cover and ESODLS algorithms?

Enumeration problems (OEIS):

- expanding current sequences
- enumerating DLS and ODLS of special kind (string-inverse, symmetric, …) and its CFs

Pseudo triples:

- 3 kinds of pseudo triples, only 1 was investigated in details

Thank you for your attention!

Thanks to all the volunteers who took part in the Gerasim@home project!

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